Special Relativity and Particle Physics Revision Problems

Rhys Davies, St. Catherine's College, 2010

***Throughout, we use units where** $c = 1$.*

1. (a) If we say that $|U\rangle$ represents an 'unstable state', we mean that if a system is in state $|U\rangle$ at time $t = 0$, then its probability of being found in state $|U\rangle$ goes to zero as $t \to \infty$. Explain why, in quantum mechanics, the energy of such a state $|U\rangle$ cannot be well-defined.

   (b) Note that when studying scattering, our initial and final states are only approximate energy eigenstates of the total Hamiltonian. We will now give a rough derivation of the Breit-Wigner resonance formula, following Cottingham and Greenwood: Let $|R\rangle$ represent a resonance — an unstable state of approximate energy E_0. We will assume that our system is in this state at time $t = 0$:

$$|\psi, 0\rangle = |R\rangle$$

   Let $\{|f\rangle\}$ be the set of all possible final states, of energy E_f. We will assume that they are orthonormal: $\langle f|f'\rangle = \delta_{f,f'}$, and expand $|\psi, t\rangle$ in terms of these states:

$$|\psi, t\rangle = a_0(t)e^{-iE_0t/\hbar}|R\rangle + \sum_f a_f(t)e^{-iE_ft/\hbar}|f\rangle$$

   Since we are dealing with approximate energy eigenstates, we have tried to factor out 'most' of the time dependence.

   i. Assuming that, for $f \neq f'$, $H_{f,f'} := \langle f|\hat{H}|f'\rangle \ll 1$, show that the Schrödinger equation gives

$$i\hbar\frac{da_f}{dt} = H_{f,R}e^{-i(E_0-E_f)t/\hbar}a_0$$

   ii. Now, since $|R\rangle$ is unstable, we will make the *ansatz*

$$a_0(t) = e^{-t/2\tau}$$

   so that the probability of the system still being in state $|R\rangle$ at time t is $e^{-t/\tau}$. Given this, show that

$$\lim_{t\to\infty} |a_f(t)|^2 = \frac{|H_{f,R}|^2}{(E_f - E_0)^2 + \hbar^2/4\tau^2}$$

   This is, of course, the probability that the final state is $|f\rangle$. This is the basic origin of the Breit-Wigner distribution.

iii. Define Γ to be the "full width at half maximum" of the above distribution, and show that

$$\Gamma = \frac{\hbar}{\tau}$$

2. (a) *A priori*, a meson of spin J might have any values of parity P and charge parity C (if it is its own anti-particle). In the quark model, such a meson is a bound state of a quark and the corresponding anti-quark. For a given value of J, which combinations of P and C therefore cannot occur in the quark model?
(Hint: The spin J of the meson is a combination of the orbital angular momentum L and spin S of the $q\overline{q}$ system. Consider the different values of S separately, and consult Martin & Shaw, section 4.4 if you're stuck.)

   (b) In the quark model the ρ^0 meson, which has spin $J = 1$, is a bound state of $u\overline{u}$ or $d\overline{d}$, like the π^0. Given that its dominant decay mode is $\rho^0 \to \pi^+\pi^-$, deduce its parity and charge parity.

   (c) Explain why the decay $\rho^0 \to \pi^0\pi^0$ cannot occur.

3. (Assume throughout this question that neutrinos are massless.)

   (a) Draw the simplest Feynman diagram describing decay of a π^- to $e^-\overline{\nu_e}$ or $\mu^-\overline{\nu_\mu}$. In each case, calculate the final energy of the charged lepton in the rest frame of the initial π^-.

   (b) Recall that a particle with spin $\vec{s}$ and momentum $\vec{p}$ is said to have *positive helicity* if $\langle \vec{s} \cdot \vec{p} \rangle > 0$ and *negative helicity* if $\langle \vec{s} \cdot \vec{p} \rangle < 0$. The amplitude for a weak decay to produce a positive helicity lepton or a negative helicity anti-lepton is proportional to $(1 - v)^{1/2}$, where v is the velocity of the (anti-)lepton. Use this fact to argue that charged pions should decay predominantly to muons rather than electrons. Neglecting phase space (density of states) factors, estimate the ratio

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu_e})}{\Gamma(\pi^- \to \mu^-\overline{\nu_\mu})}$$

   (The empirical value is $\sim 10^{-4}$).

4. (a) Write down the Lorentz transformations for energy E and momentum $\mathbf{p}$. Show that the quantity $E^2 - |\mathbf{p}|^2 c^2$ is invariant. Now consider two particles A and B with energies E_{A} and E_{B} and momenta $\mathbf{p}_{\mathrm{A}}$ and $\mathbf{p}_{\mathrm{B}}$, respectively. What can you say about the quantity $E_{\mathrm{A}} E_{\mathrm{B}} - \mathbf{p}_{\mathrm{A}} \cdot \mathbf{p}_{\mathrm{B}}\, c^2$?

(b) Particle Y of mass m_{Y} decays at rest into particles A and C with masses m_{A} and m_{C}, respectively.

i. Derive an expression for the energy of particle C in the lab frame in terms of the particle masses.

ii. Now consider a three-body decay of particle Y at rest into products A, B (of mass m_{B}) and C, all of which have non-zero rest mass. By considering A and B as a composite particle X, or otherwise, show that the energy of C in the lab frame is

$$E_{\mathrm{C}} = \frac{(m_{\mathrm{Y}}^2 + m_{\mathrm{C}}^2 - m_{\mathrm{A}}^2 - m_{\mathrm{B}}^2)\, c^4 - 2E_{\mathrm{A}} E_{\mathrm{B}} + 2\mathbf{p}_{\mathrm{A}} \cdot \mathbf{p}_{\mathrm{B}}\, c^2}{2 m_{\mathrm{Y}} c^2}.$$

Give an expression for the maximum energy of particle C.

iii. By taking the limit as $m_{\mathrm{B}} \to 0$, find an expression for the maximum energy of particle C when particle B is massless.

The trouble with the way the answer to part (iii) was obtained is that massless particles may only travel at the speed of light. Comment on why the answer to part (iii) is nevertheless correct.