Quotients of the conifold in compact Calabi-Yau varieties

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Outline

Motivation

Multiply-connected Calabi-Yau manifolds

Hyperconifold singularities

Hyperconifolds and toric geometry

Resolving singularities: hyperconifold transitions

Outline

Heterotic string compactification

There are two approaches to model-building in heterotic string theory:

1. Exact CFTs

   Solvable worldsheet theories e.g. toroidal orbifolds, free fermions etc.

2. Geometric compactifications

   Solutions of 10D supergravity, with stringy corrections.

Both can yield realistic gauge groups and spectra.

Smooth Calabi-Yau compactifications

A smooth geometric compactification of $E_8 \times E_8$ heterotic string theory:

- A Calabi-Yau threefold X (Kähler, $c_1(X) = 0$).
- A (stable, holomorphic) vector bundle V on X. The gauge field is a connection on this bundle.

This amounts to a solution of the Einstein-Yang-Mills equations on X, preserving minimal supersymmetry in 4D.

We usually assume the standard model comes entirely from one E_8.

Observable gauge group

Background gauge field takes values in a subgroup $H \subset E_8$, called the *structure group* of the vector bundle. Resulting 4D gauge group is the *centraliser* of H in E_8.

Structure group	$SU(3)$	$SU(4)$	$SU(5)$
4D gauge group	E_6	Spin(10)	$SU(5)$

Problem: There are no Higgs fields present which can break these GUT groups to the standard model gauge group.

Wilson line symmetry breaking

Solution: If spacetime is not simply-connected, the field strength (curvature) does not specify the gauge field. Complete information contained in Wilson loops:

$$W(\gamma) = \mathcal{P} \exp(\int_\gamma A)$$

Break the 4D gauge group further by turning on Wilson loops around non-contractible paths.

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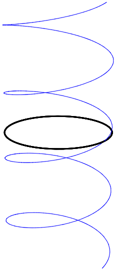
Covering spaces

A multiply-connected manifold has a unique simply-connected covering space.

Example:

The circle S^1, covered by the real line $\mathbb{R}$.



The manifold is then a quotient of its covering space.

The $\mathbb{Z}_5$ quotient of the quintic

A well-known Calabi-Yau 3-fold is a quintic hypersurface $\widetilde{X}$ in $\mathbb{P}^4$:

$$p := \sum A_{ijklm} \, x_i \, x_j \, x_k \, x_l \, x_m = 0$$

Define an action of the group $\mathbb{Z}_5$ on $\mathbb{P}^4$:

$$(x_0, x_1, x_2, x_3, x_4) \to (x_0, \zeta x_1, \zeta^2 x_2, \zeta^3 x_3, \zeta^4 x_4) \quad \text{where} \quad \zeta = \exp(2\pi \mathrm{i}/5)$$

Invariant quintic hypersurface:

$$A_{ijklm} = 0 \quad \text{unless} \quad i + j + k + l + m \equiv 0 \mod 5.$$

The $\mathbb{Z}_5$ quotient of the quintic

The $\mathbb{Z}_5$ quotient is smooth if and only if:

- The group action is fixed-point-free.
- The covering space (given by $p = 0$) is smooth.

Checking fixed points

$(1, 0, 0, 0, 0) \in \mathbb{P}^4$ is fixed under $\mathbb{Z}_5$. At this point $p = A_{00000} \neq 0$.

Checking smoothness

The quintic is smooth if $p = dp = 0$ has no solutions.

$X = \widetilde{X}/\mathbb{Z}_5$ is thus a smooth Calabi-Yau manifold with $\pi_1(X) \simeq \mathbb{Z}_5$.

Many more examples: Candelas & Davies **arXiv:0809.4681**

Fixed points

In many cases, fixed points arise automatically.

The resulting quotient spaces are not manifolds, but *orbifolds*.

Orbifolds

An orbifold here is locally $\mathbb{C}^3/G$ for some group G (including trivial).

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Demanding fixed points

Special choice $A_{00000} = 0$ gives $p = 0$ at fixed point $(1, 0, 0, 0, 0)$.

In local coordinates $y_a = x_a/x_0$,

$$p = y_1\, y_4 - y_2\, y_3 + \mathcal{O}(y^3)$$

So in fact $p = dp = 0$ at the fixed point, and $\widetilde{X}$ has a conifold singularity.

X develops a worse local singularity: a $\mathbb{Z}_5$ quotient of the conifold.
Call this a *hyperconifold* singularity.

Proof of occurrence of hyperconifolds

Question: Is the quintic example an accident?

<u>Scenario</u>

- $\mathbb{Z}_N$ acts on $\mathbb{C}^{k+3}$ via $x_i \rightarrow \zeta^{q_i} x_i$, where $\zeta = \exp(2\pi \mathrm{i}/N)$.
- $q_1 = \ldots = q_{\dim I} = 0$; I is subspace of points fixed by $\mathbb{Z}_N$.
- $\widetilde{X}$ given locally by k equations $f_1 = \ldots = f_k = 0$ in $\mathbb{C}^{k+3}$.
- Polynomials transform as $f_a \rightarrow \zeta^{Q_a} f_a$.
- For generic choices of the f_a, $\mathbb{Z}_N$ action on $\widetilde{X}$ is free.

Proof of occurrence of hyperconifolds

Rough proof:

1. $f_a|_I \equiv 0$ unless $Q_a = 0$.
2. Action on $\widetilde{X}$ free $\Rightarrow f_1 = \ldots f_k = 0$ no solutions on I.
3. 1. and 2. imply $Q_1 = \ldots = Q_{\dim I+1} = 0$.
4. Enforce a fixed point: set $f_a = 0$ for all a at origin.
5. If $1 \leq a \leq \dim I + 1$, we expand

$$f_a = \sum_{i=1}^{\dim I} C_{a,i}\, x_i + \mathcal{O}(x^2)$$

6. Then $df_1 \wedge \ldots \wedge df_{\dim I+1} = 0$ at origin, so $\widetilde{X}$ is singular there.

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The conifold

The conifold as a hypersurface

Simplest singularity of a complex threefold:

$$y_1 \, y_4 - y_2 \, y_3 = 0 \quad \text{in } \mathbb{C}^4$$

Two ways to think of the topology:

- A complex cone over $S^2 \times S^2$ ($\mathbb{P}^1 \times \mathbb{P}^1$).
- A real cone over $S^2 \times S^3$.

The conifold

The conifold as a quotient

Four coordinates (z_1, z_2, z_3, z_4) on $\mathbb{C}^4 \setminus \mathcal{S}$.

$$(z_1, z_2, z_3, z_4) \sim (\lambda\, z_1, \lambda\, z_2, \lambda^{-1}\, z_3, \lambda^{-1}\, z_4) \quad \text{for all } \lambda \in \mathbb{C}^*$$

Call these the 'homogeneous coordinates'. Isomorphism to hypersurface:

$$y_1 = z_1 z_3 \ , \quad y_2 = z_1 z_4 \ , \quad y_3 = z_2 z_3 \ , \quad y_4 = z_2 z_4$$

Toric varieties

An n-dimensional toric variety is an algebraic variety Y which

- Contains $(\mathbb{C}^*)^n$ as a dense subset.
- Admits an action $(\mathbb{C}^*)^n \times Y \to Y$ extending the action of $(\mathbb{C}^*)^n$ on itself.

Specified by a *fan* in the lattice $N \simeq \mathbb{Z}^n$.

A simple toric variety: $\mathbb{P}^2$

The torus $(\mathbb{C}^*)^2$ is embedded in $\mathbb{P}^2$:

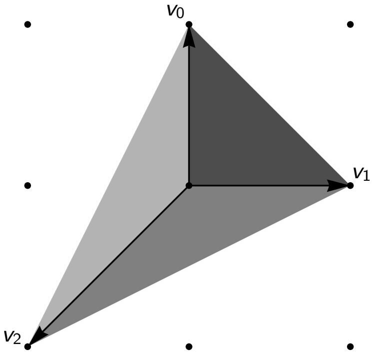
$$(\lambda_1, \lambda_2) \to (1, \lambda_1, \lambda_2)$$

and acts on it appropriately:

$$(\lambda_1, \lambda_2) \cdot (x_0, x_1, x_2) = (x_0, \lambda_1 x_1, \lambda_2 x_2)$$

The fan for $\mathbb{P}^2$

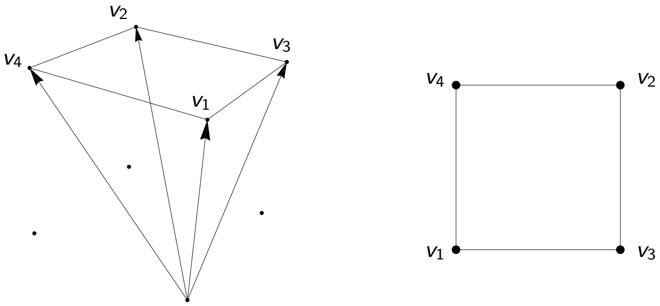
The edges of the fan are generated by $v_0, v_1, v_2 \in \mathbb{Z}^2$ satisfying $v_0 + v_1 + v_2 = 0$



Corresponds to $(x_0, x_1, x_2) \sim (\lambda x_0, \lambda x_1, \lambda x_2)$.

The conifold as a toric variety

Consider edges generated by $v_1, v_2, v_3, v_4 \in \mathbb{Z}^3$ with $v_1 + v_2 - v_3 - v_4 = 0$:



Corresponds to $(z_1, z_2, z_3, z_4) \sim (\lambda\, z_1, \lambda\, z_2, \lambda^{-1} z_3, \lambda^{-1} z_4)$.

Some toric geometry facts

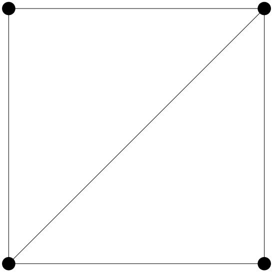
Let Y be an n-dimensional toric variety.

1. Y has at most orbifold singularities (*i.e.* $\mathbb{C}^n/G$ for some discrete group G) iff its fan contains only simplicial cones.
2. Y is non-singular iff all cones are simplicial of minimal volume.

For 2, each cone is just $\langle(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\rangle$. This is simply $\mathbb{C}^n$.

Resolving the conifold

Sub-divide the fan to obtain a smooth (crepant) resolution:



The singular point has been replaced by a copy of $\mathbb{P}^1$.

Doesn't necessarily give a Kähler resolution of the compact variety.

Revisiting the $\mathbb{Z}_5$-hyperconifold

Recall for the $\mathbb{Z}_5$ quintic we get:

$$\{y_1\, y_2 - y_3\, y_4 = 0\}/\sim \quad \text{where} \quad (y_1, y_2, y_3, y_4) \sim (\zeta\, y_1, \zeta^2\, y_2, \zeta^3\, y_3, \zeta^4\, y_4)$$

But the y_a are given in terms of the homogeneous coordinates

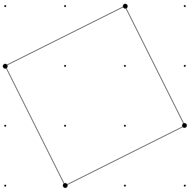
$$y_1 = z_1 z_3\ , \quad y_2 = z_1 z_4\ , \quad y_3 = z_2 z_3\ , \quad y_4 = z_2 z_4$$

So we get an extra equivalence relation on the homogeneous coordinates:

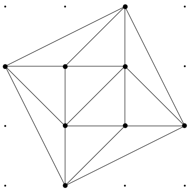
$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^2 z_2, \zeta z_3, \zeta^2 z_4)$$

The fan for the $\mathbb{Z}_5$-hyperconifold

Discrete factors in quotient group give the same fan in a new lattice:



The toric formalism makes it easy to resolve the singularity:



Important example: The $\mathbb{Z}_2$-hyperconifold

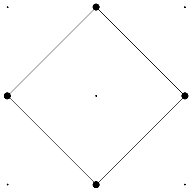
Unique $\mathbb{Z}_2$ action fixing only the origin:

$$(y_1, y_2, y_3, y_4) \to (-y_1, -y_2, -y_3, -y_4)$$

Resulting equivalence relation on homogeneous coordinates:

$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, -z_3, -z_4)$$

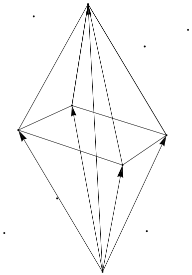
The fan is now



Outline

Blowing up the conifold

Guarantee a Kähler resolution of the conifold by ‘blowing up’

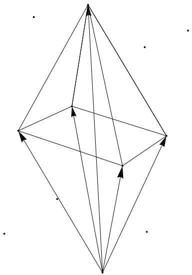


New vector lies out of hyperplane $\Rightarrow$ not a crepant resolution.

Variety with one conifold singularity has no Calabi-Yau resolution.

Blowing up the conifold

Guarantee a Kähler resolution of the conifold by 'blowing up'



This manifold is given explicitly by

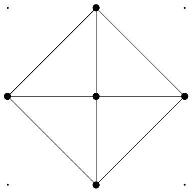
$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda\, z_1, \lambda\, z_2, \mu\, z_3, \mu\, z_4, \lambda^{-1}\mu^{-1} z_5)$$

This is the bundle $\mathcal{O}(-1,-1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$.

Blowing up the $\mathbb{Z}_2$-hyperconifold

Story for $\mathbb{Z}_2$ quotient is different, due to different lattice.

Blowing up now gives a crepant resolution



This space is given by

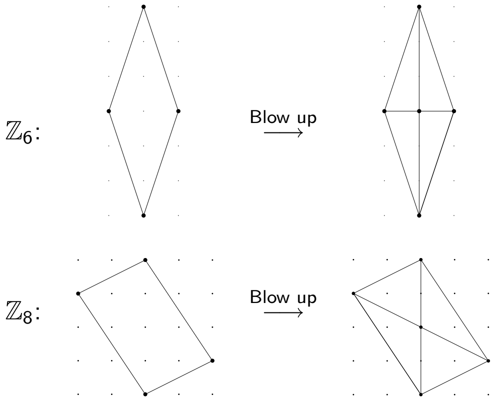
$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda\, z_1, \lambda\, z_2, \mu\, z_3, \mu\, z_4, \lambda^{-2}\mu^{-2} z_5)$$

This is $\mathcal{O}(-2,-2)$ over $\mathbb{P}^1 \times \mathbb{P}^1$.

The $\mathbb{Z}_{2M}$-hyperconifolds

The above analysis can be carried out for all known $\mathbb{Z}_N$ actions.

For $N = 2M$, we can blow up the singular point as before:



This leaves only orbifold singularities with unique resolutions.

Hyperconifold transitions in string theory?

Summary of process:

- Begin with smooth Calabi-Yau X.
- Deform until a hyperconifold singularity develops.
- Blow up singularity.

This is a continuous path through Calabi-Yau moduli space.

Perhaps also a continuous process in string theory.

New light degrees of freedom: winding modes/twisted sectors of strings.

Summary

- Multiply-connected Calabi-Yau threefolds generically develop isolated ‘hyperconifold’ singularities.
- This lets us explicitly embed hyperconifolds in compact Calabi-Yau varieties.
- Using toric geometry, such singularities can be resolved to yield new smooth Calabi-Yau manifolds.