Notes on the del Pezzo surface of degree six

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This is an informal set of notes on the geometry of the del Pezzo surface of degree six, which we will denote by dP_6. This is the surface obtained by blowing up the (complex) projective plane at three generic points. Note that this surface is often referred to as dP_3 in the physics literature.

Generalities

The del Pezzo surfaces are, by definition, the smooth algebraic surfaces S with ample anti-canonical divisor class K_S. There are a finite number of topological types of such surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown up at $9-d$ generic points, where $d = 0, \ldots, 8$. The degree d is defined as the self-intersection of the anti-canonical divisor class: $d(S) = K_S \cdot K_S$. It is easy to check that for the blow-ups of $\mathbb{P}^2$, the degree is the same d defined above. The degree of $\mathbb{P}^1 \times \mathbb{P}^1$ is 8.

The surface dP_6 is defined by blowing up the complex projective plane, $\mathbb{P}^2$, at three generic points ('generic' meaning that they don't lie on a line) p_1, p_2, p_3. By choice of coordinates we can assume that these points lie at $(1,0,0), (0,1,0), (0,0,1)$, which we will do throughout. Note that this implies that dP_6 is rigid — it has no deformations.

As usual, each blow-up introduces an exceptional curve, isomorphic to $\mathbb{P}^1$, with self-intersection -1; these three curves will be denoted by E_1, E_2 and E_3 respectively. Together with H, the proper transform of a generic hyperplane in $\mathbb{P}^2$, these span $H_2(\mathrm{dP}_6, \mathbb{Z})$.

It is straightforward to work out the intersection form on dP_6. We started with $\mathbb{P}^2$, which is a smooth surface, and blew up three distinct points, giving exceptional curves E_1, E_2, E_3. It is a general fact that such a curve satisfies $E \cdot E = -1$, so because our three curves are disjoint, we get $E_i \cdot E_j = -\delta_{ij}$. For intersections involving H, note that we can always choose hyperplanes which miss the three blown-up points. Altogether, we get

$$H \cdot H = 1 \ , \quad E_i \cdot E_j = -\delta_{ij} \ , \quad H \cdot E_i = 0 \tag{1}$$

There are three more important lines on dP_6. Let L_i denote the proper transform of the unique line in $\mathbb{P}^2$ which contains the points p_{i+1} and p_{i+2} (with the subscripts understood modulo 3). Since any two distinct hyperplanes in $\mathbb{P}^2$ intersect transversely at a single point, and L_{ij} also intersects E_i and E_j at one point each, we conclude from inspection of the intersection form above that

$$L_i = H - E_{i+1} - E_{i+2}$$

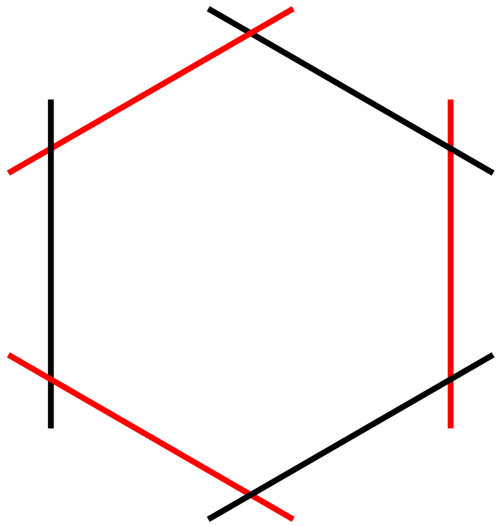

Figure 1: The hexagon formed by the six -1-lines in dP_6. The black lines are the E_i, and the red lines the L_i. The symmetry group D_6 of the hexagon actually acts on dP_6.

From this we can easily check that $L_i \cdot L_j = -\delta_{ij}$, and in fact, the L_i are equivalent to the E_i, and together they form a hexagon, as shown in Figure 1. It's clear from the way we have constructed it that dP_6 is simply connected, since $\mathbb{P}^2$ is, and blowing up points does not introduce any homotopically non-trivial paths. It is also straightforward to deduce the Euler characteristic. $\mathbb{P}^2$ has Euler characteristic $\chi(\mathbb{P}^2) = 3$, and each blow-up replaces a point, $\chi(\mathrm{pt}) = 1$, with a two-sphere, $\chi(S^2) = 2$, so we have $\Delta\chi = 3$, and therefore $\chi(\mathrm{dP}_6) = 6$.

As a complete intersection

There are several ways to write explicit equations for dP_6 as a complete intersection in a product of projective spaces. Perhaps the most useful is to take the ambient space to be $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$; if we take homogeneous coordinates $(t^{i0}, t^{i1}), i = 0, 1, 2$ on the three $\mathbb{P}^1$'s, and (x^0, x^1, x^2) on $\mathbb{P}^2$, then the equations can be taken to be

$$\begin{aligned} x^1\, t^{00} - x^2\, t^{01} &= 0 \\ x^2\, t^{10} - x^0\, t^{11} &= 0 \\ x^0\, t^{20} - x^1\, t^{21} &= 0 \end{aligned} \tag{2}$$

It is easy to see, for example, that when $x^1 = x^2 = 0$, (t^{00}, t^{10}) is left completely undetermined, so over this point of $\mathbb{P}^2$, we get an entire copy of $\mathbb{P}^1$, which is the exceptional curve E_1.

Writing dP_6 is this way gives us a convenient representation of its Picard group. Each projective space comes with the distinguished line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$, and the Picard group of dP_6 is spanned by the pullbacks of these bundles from each of the four projective spaces which make up the ambient space in which we impose (2).

We can equivalently write (2) as a matrix equation

$$M \cdot x \;:=\; \begin{pmatrix} 0 & t^{00} & -t^{01} \\ -t^{11} & 0 & t^{10} \\ t^{20} & -t^{21} & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Since the x's cannot all vanish simultaneously, there exist solutions only when $\det M = 0$. This is equivalent to the condition that M has rank less than three. Clearly M cannot have rank zero, since we cannot have $t^{i0} = t^{i1} = 0$, but can it have rank one? This is the case if and only if the second and third rows are multiples of the first. The second row is a multiple of the first exactly when $t^{00} = t^{11} = 0$, and the third row is a multiple of the first exactly when $t^{01} = t^{20} = 0$. Obviously these cannot both be true, so we conclude that $\det M = 0 \iff \mathrm{rank}(M) = 2$. What we have demonstrated is that, given a point in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ such that $\det M = 0$, there is a unique point of $\mathbb{P}^2$ which solves equation (3). So the x's contain no extra information, and we can equally well specify dP_6 by a single trilinear equation in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$:

$$\det M \;=\; t^{00}t^{10}t^{20} - t^{01}t^{11}t^{21} \;=\; 0 \tag{4}$$

We may also represent dP_6 as a complete intersection of two bilinear equations in $\mathbb{P}^2\times\mathbb{P}^2$, but I will not describe this here.

Line bundles

There is a well-known correspondence between divisors and line bundles. We have described a set of independent generators $\{H, E_1, E_2, E_3\}$ for the group of divisor classes, and also a basis for the Picard group, given by the pullbacks of the bundles $\mathcal{O}(1)$ from each projective space when we embed dP_6 in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$. We will denote an arbitrary tensor product of powers of such bundles, in an obvious notation, by $\mathcal{O}(a, b, c, d)$. It would be useful to relate these to the bundles obtained from our generating set of divisors; $\mathcal{O}(H)$ and so on.

Consider first the bundle $\mathcal{O}(1, 0, 0, 0)$ i.e. the pullback of $\mathcal{O}(1)$ on the first $\mathbb{P}^1$. Take a generic section of this bundle: $\sigma = t^{00} - at^{01}$, where a is a constant. Then substituting $\sigma = 0$ into (2) specifies a curve C inside dP_6, given by

$$\begin{aligned} ax^1 - x^2 &= 0 \\ x^2 t^{10} - x^0 t^{11} &= 0 \\ x^0 t^{20} - x^1 t^{21} &= 0 \end{aligned} \tag{5}$$

We now want to work out the intersection of C with H and the E_i. First of all, it is clear that if we impose another linear condition on the x's, then there is a unique solution to (5), so $C \cdot H = 1$. Now consider $C \cdot E_1$. E_1 is given by $x^1 = x^2 = 0$. This identically satisfies the first equation in equation (5), and, since necessarily $x^0 \neq 0$, the last two simply become $t^{11} = t^{20} = 0$, so we have a unique solution, and therefore $C \cdot E_1 = 1$. On the other hand, E_2 is given by $x^0 = x^2 = 0$, and the first equation of equation (5) then forces $x^1 = 0$, which is impossible, so $C \cdot E_2 = 0$. The same reasoning gives $C \cdot E_3 = 0$. These intersection numbers uniquely determine that $C \sim H - E_1$, and therefore $\mathcal{O}(1,0,0,0) \cong \mathcal{O}(H - E_1)$. Similar calculations give us the full identification between the line bundles and divisors we have encountered:

$$\begin{aligned} \mathcal{O}(1,0,0,0) &\cong \mathcal{O}(H - E_1)\ ,\ \ \mathcal{O}(0,1,0,0) \cong \mathcal{O}(H - E_2) \\ \mathcal{O}(0,0,1,0) &\cong \mathcal{O}(H - E_3)\ ,\ \ \mathcal{O}(0,0,0,1) \cong \mathcal{O}(H) \end{aligned}$$

As a toric variety

$\mathbb{P}^2$ is toric, with the torus embedding conveniently given by

$$(x^0, x^1, x^2) = (1, \tau^1, \tau^2)\ ,\quad (\tau^1, \tau^2) \in (\mathbb{C}^*)^2 \tag{6}$$

The three points we blow up to obtain dP_6 are the fixed points of the toric action, and therefore dP_6 is also a toric surface. Indeed, the torus embedding of equation (6) can easily be extended to

$$\begin{aligned} (x^0, x^1, x^2) &= (1, \tau^1, \tau^2) \\ (t^{00}, t^{01}) &= (\tau^2, \tau^1) \\ (t^{10}, t^{11}) &= (1, \tau^2) \\ (t^{20}, t^{21}) &= (\tau^1, 1) \end{aligned} \tag{7}$$

These replacements identically satisfy the equations in (2), so indeed this is an embedding of the torus in dP^6. We can obtain the fan for dP_6 by the usual prescription for blow-ups, of starting with the fan for $\mathbb{P}^2$ and sub-dividing the cones corresponding to the blown-up points. The result is shown in Figure 2. The toric description also immediately confirms our earlier statements about topology. Any toric variety whose fan contains a cone of maximal dimension is simply-connected, and the Euler characteristic is equal to the number of top-dimensional cones, which here is six.

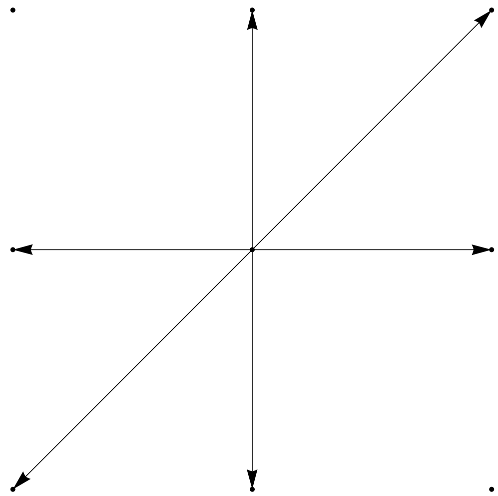

Figure 2: The fan for dP_6, considered as a toric surface.

It is easy to check that the six toric divisors, given by the six one-dimensional cones in Figure 2, correspond, reading clockwise, to the six curves $(E_1, L_3, E_2, L_1, E_3, L_2)$.

Hodge numbers

We can actually use indirect arguments to calculate the Hodge numbers of dP_6 from what we already know. It is clear that dP_6 admits Kähler metrics, since it can be embedded in a product of projective spaces, so the Hodge numbers satisfy

$$h^{p,q} = h^{q,p} = h^{2-p,2-q}$$

Together with the knowledge that the Euler number is 6, this gives

$$2h^{0,0} - 4h^{0,1} + 2h^{0,2} + h^{1,1} = 6$$

We also know that the surface is connected and simply-connected, so $h^{0,0} = 1$ and $h^{0,1} = 0$, and we get

$$h^{1,1} = 4 - 2h^{0,2}$$

Since we have exhibited four independent divisors $\{H, E_1, E_2, E_3\}$, whose Chern classes are therefore independent cohomology classes of type $(1,1)$, it must be true that $h^{1,1} \geq 4$, so we conclude that in fact $h^{1,1} = 4$, $h^{0,2} = 0$. Altogether, the Hodge diamond for dP_6 is given by

$$\begin{array}{ccccc} & & h^{00} & & \\ & h^{10} & & h^{01} & \\ h^{20} & & h^{11} & & h^{02} \\ & h^{21} & & h^{12} & \\ & & h^{22} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 4 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$