Notation and summary

We will consider heterotic/F-theory duals; here we establish the relevant notation once and for all. It is a combination of notation from Friedman, Morgan, and Witten, and from Hayashi et al.

Heterotic side

The ingredients on the heterotic side which give a single $SU(n)$ bundle:

- A Calabi–Yau N-fold Z, elliptically-fibred over S, with projection $\pi_Z : Z \to S$, and section $\sigma : S \to Z$.
- The spectral cover C, which is an n-fold branched cover of S, $\pi_C : C \to S$. Considered as a sub-variety of Z, C intersects the section σ along a divisor $nK_S + \eta$.
- The spectral line bundle $\mathcal{N}_V$ on C.
- An auxiliary object is the fibre product $C \times_S Z$, with projections p_C and p_Z to the two factors.
- The gauge bundle V is constructed from C and $\mathcal{N}_V$ via Fourier-Mukai transform. In particular, if $\mathcal{P}$ is the Poincaré line bundle on $Z \times Z$, then V is given by

$$V = p_{2*}(p_1^* \mathcal{N}_V \otimes \mathcal{P}) \ .$$

F-theory side

On the F-theory side we have the following:

- A Calabi–Yau $(N+1)$-fold X, elliptically-fibred over an N-fold B.
- In turn, B is a $\mathbb{P}^1$ fibration over S; in fact, $B = \mathbb{P}\big(\mathcal{O}_S \oplus \mathcal{O}_S(6K_S + \eta)\big)$, where η is the same class used to construct the spectral cover above. If we take into account both E_8 factors on the heterotic side, which correspond to the poles of the $\mathbb{P}^1$ fibre, we find the relation $\eta_0 + \eta_1 = -12K_S$. This makes the construction consistent, since

$$\mathbb{P}\big(\mathcal{O}_S \oplus \mathcal{O}_S(6K_S + \eta_0)\big) \cong \mathbb{P}\big(\mathcal{O}_S \oplus \mathcal{O}_S(-6K_S - \eta_1)\big) \cong \mathbb{P}\big(\mathcal{O}_S \oplus \mathcal{O}_S(6K_S + \eta_1)\big) \ .$$

The basic duality between the heterotic string and F-theory states that the heterotic string compactified on a torus is equivalent to F-theory compactified on an elliptic $K3$ surface, which can be interpreted as type IIB string theory compactified on $\mathbb{P}^1$ with a varying axio-dilaton. This basic duality can then be applied fibrewise to obtain dualities between the theories compactified to lower dimensions. These notes are intended to flesh out this idea, largely following the paper by Friedman, Morgan, and Witten.

1 Elliptically-fibred Calabi–Yau manifolds

Since elliptically-fibred Calabi–Yau manifolds appear on both sides of the F-theory/heterotic duality, we will begin by explaining how they are constructed.

Let $\pi : X \to B$ be an elliptic fibration, where X is a Calabi–Yau manifold. The term *elliptic* fibration means that there exists at least one section $\sigma : B \hookrightarrow X$, and for each $b \in B$, we take $\sigma(b)$ to be the zero of the group law on $\pi^{-1}(b)$.

The existence of a section means that a minimal model of X can be described by a Weierstrass model,

$$Y^2 Z = X^3 + fXZ^2 + gZ^3 \ , \tag{1}$$

where the parameters can vary over the base B. The section is given by $Z = 0$ (the above equation then implies $X = 0$). We can now ask under what circumstances such a model does in fact describe a Calabi–Yau. Without loss of generality, we can assume that X, Y, Z are the homogeneous coordinates on a $\mathbb{P}^2$ bundle of the form $P = \mathbb{P}\big(\mathcal{O}_B(2D) \oplus \mathcal{O}_B(3D) \oplus \mathcal{O}_B\big)$ for some divisor D (an overall twist makes no difference, so Z can be made a section of $\mathcal{O}_B$, then note that X^3 and Y^2 must be sections of the same bundle for the equation to be well-defined). If we denote by H the divisor class which corresponds to the hyperplane class on each fibre, we see that the Weierstrass equation is a section of $\mathcal{O}_P(3H + 6\,\pi^* D)$. Therefore for (1) to define a Calabi–Yau, we require

$$-K_P \sim 3H + 6\,\pi^* D \tag{2}$$

To find an expression for $-K_P = c_1(P)$, consider first the Euler sequence on $\mathbb{P}^2$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \longrightarrow T\mathbb{P}^2 \longrightarrow 0 \ .$$

The three summands of the middle term correspond to the homogeneous coordinates (X, Y, Z). Therefore, if we let TF be the sub-bundle of TP which corresponds to vectors pointing along the fibres, we can fibre this sequence over B to obtain

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P(2\,\pi^* D + H) \oplus \mathcal{O}_P(3\,\pi^* D + H) \oplus \mathcal{O}_P(H) \longrightarrow TF \longrightarrow 0 \ .$$

We also have the short exact sequence

$$0 \longrightarrow TF \longrightarrow TP \longrightarrow \pi^* TB \longrightarrow 0 \ ,$$

and if we put these together (and use $-K \sim c_1$) we find

$$-K_P \sim 3H + \pi^*(-K_B + 5\,\pi^* D) \ .$$

Comparing to (2), we see that our Weierstrass model describes a Calabi–Yau if we choose $D \sim -K_B$. The parameters f and g are therefore sections of ω_B^{-4} and ω_B^{-6} respectively.

Note that B is embedded inside X by

1.1 In weighted bundles

We can alternatively form the projective bundle with non-trivial weights, i.e., take $P = \mathbb{WP}_{1,2,3}\big(\mathcal{O}_B \oplus \mathcal{O}_B(2D) \oplus \mathcal{O}_B(3D)\big)$. With corresponding homogeneous coordinates (V, X, Y), the Weierstrass equation takes the same form:

$$Y^2 = X^3 + fXV^2 + gV^6 \ .$$

Using the same argument as above, we again find that the Calabi–Yau condition implies $D \sim -K_B$.

2 The eight-dimensional duality

We start by outlining the equivalence between the heterotic string compactified on a torus, and F-theory compactified on a $K3$ surface.

2.1 Flat $SU(n)$ bundles on a torus — explicit construction

When the heterotic string is compactified on a torus, T-duality means that there is no general way to distinguish geometric parameters from bundle parameters. However, if we fix the Kähler modulus of the torus to be very large, and choose a fixed complex structure, we can isolate a "bundle moduli space". The equations of motion simply say that the bundle must be flat.

We will focus on one E_8 of the $E_8 \times E_8$ heterotic string, and on gauge bundles with structure group $SU(n)$. A flat bundle on an elliptic curve $\mathcal{E}$ is just a sum of degree-zero line bundles, so we must understand these.

Let p_0 be the zero of the group law on $\mathcal{E}$. Then any degree-zero line bundle is of the form $\mathcal{O}(P) \otimes \mathcal{O}(p_0)^{-1}$ for some point $P \in \mathcal{E}$; we will denote this bundle by $\mathcal{L}(P)$. Let $\phi : \mathcal{E} \hookrightarrow \mathbb{P}^2$ be the embedding given by the linear system $|3p_0|$. Then by the definition of the group law, $P + Q + R = 0$ if and only if $\mathcal{O}(P) \otimes \mathcal{O}(Q) \otimes \mathcal{O}(R) \cong \phi^* \mathcal{O}(1) \cong \mathcal{O}(p_0)^3$. Therefore $\mathcal{L}(P) \otimes \mathcal{L}(Q) \otimes \mathcal{L}(R) \cong \mathcal{O}$ if and only if $P + Q + R = 0$.

So a flat $SU(n)$ bundle on $\mathcal{E}$ is of the form $\bigoplus_{i=1}^n \mathcal{L}(Q_i)$, where $\sum_{i=1}^n Q_i = 0$ in the group law on $\mathcal{E}$. The isomorphism $\bigotimes_{i=1}^n \mathcal{L}(Q_i) \cong \mathcal{O}$ is equivalent to the statement that the divisor $(n\, p_0 - \sum_{i=1}^n Q_i)$ is principal; there is a unique meromorphic function on $\mathcal{E}$ with simple zeros at the Q_i and an n^{th} order pole at p_0. We will give an explicit description of such functions.

Let $\mathcal{E}$ be given by the Weierstrass equation

$$Y^2 Z = X^3 + fXZ^2 + gZ^3 \ . \tag{3}$$

Then the point p_0 is at $[X : Y : Z] = [0 : 1 : 0]$. A meromorphic function with a pole at p_0 is therefore simply a polynomial in the affine coordinates $x = X/Z, y = Y/Z$. To make further progress, we must consider x and y as meromorphic functions on $\mathcal{E}$, and ask the order of their poles at p_0. To find this, define affine coordinates $s = X/Y, t = Z/Y$, so our equation becomes

$$t = s^3 + fst^2 + gt^3 \ .$$

Since p_0 is at $t = s = 0$, if w is a local coordinate on $\mathcal{E}$ satisfying $w(p_0) = 0$, then we must have $t = \mathcal{O}(w), s = \mathcal{O}(w)$. But then the right-hand side of the above equation is of order

w^3, implying that t is also of this order. Therefore s has a simple zero at p_0, while t has a zero of order 3. Then, since $x = s/t$ and $y = 1/t$, we conclude that at p_0, x has a pole of order 2, and y has a pole of order 3. We can therefore represent the n points $\{Q_i\}$ by the vanishing of a meromorphic function on $\mathcal{E}$ as follows

$$0 = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + \ldots + \begin{cases} a_n x^{\frac{n}{2}} & n \text{ even,} \\ a_n x^{(n-3)/2} y & n \text{ odd.} \end{cases} \tag{4}$$

A succint way of summarising the above is to say that the moduli space of flat $SU(n)$ bundles on $\mathcal{E}$ is $\mathbb{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(n)) \cong \mathbb{P}^{n-1}$.

Aside: The preceding argument is correct, but it may not be obvious that the equations (3) and (4) have exactly n simultaneous solutions. To see this explicitly, solve (4) for y as a rational function of x, substitute the solution into (3), and multiply through by the denominator of the y^2 term. It is easy to check that one always obtains an order n polynomial in x.

2.2 Flat $SU(n)$ bundles on a torus — del Pezzo construction

Friedman, Morgan, and Witten give an alternative description of the above moduli space in terms of the moduli of a surface M which is a del Pezzo surface of degree one. Such a surface can be described as $\mathbb{P}^2$ blown up in eight points, but for the present purpose, it is more convenient to describe it as a hypersurface in a weighted projective space:

$$M \leftrightarrow \mathbb{WP}_{1,1,2,3}[6] \ .$$

We take homogeneous coordinates (u, v, x, y) for $\mathbb{WP}_{1,1,2,3}$, corresponding to the weights $(1, 1, 2, 3)$, so that the equation for M takes the form

$$p_{\mathcal{E}}(v, x, y) + u\big(\alpha_5(u, v) + \alpha_3(u, v)x + \alpha_2(u, v)y + \alpha_1(u, v)x^2 + \alpha_0 xy\big) = 0 \tag{5}$$

where the functions α_k are homogeneous of degree k, and $p_{\mathcal{E}}$ is the Weierstrass polynomial for an elliptic curve $\mathcal{E}$ in the space $\mathbb{WP}_{1,2,3}$ defined by $u = 0$:

$$p_{\mathcal{E}}(v, x, y) = -y^2 + x^3 + f\, x\, v^4 + g\, v^6 \ .$$

The lattice of divisor classes on $M \cong \mathrm{dP}_1$ which satisfy $D \cdot K_M = 0$ is isomorphic to the root lattice of E_8. Since $\mathcal{E} \sim -K_M$, each such class defines a degree zero line bundle on $\mathcal{E}$, and in this way we obtain a flat E_8 bundle on $\mathcal{E}$.

If C is a *curve* satisfying $C \cdot K_M = 0$, then it must be disjoint from $\mathcal{E}$, and therefore define a trivial bundle. Note that by adjunction we have

$$K_C \sim (K_M + C)\big|_C \sim C\big|_C \ ,$$

and therefore in particular, $\deg K_C = C \cdot C$. Rational curves are therefore (-2)-curves, and the sub-lattice they generate corresponds to a sub-algebra of $\mathfrak{e}_8$ which is represented trivially in the bundle on $\mathcal{E}$. So the singularity in the minimal model of M corresponds to the sub-algebra of $\mathfrak{e}_8$ left unbroken by the bundle on $\mathcal{E}$.

2.3 dP_1? What has that to do with $K3$?

The del Pezzo construction has encoded the heterotic curve and one E_8 bundle in the geometry of a surface; this is the basic idea of the F-theory dual. Here we describe how the incorporation of the other E_8 factor leads to the F-theory $K3$ surface.

Note from (5) that the point $u = v = 0$ is actually a base point of the anti-canonical system of M (the anti-canonical class of dP_1 is ample but not very ample). Blow up this point to a whole $\mathbb{P}^1$ parametrised by $[u : v]$; the projection to this $\mathbb{P}^1$ makes the resulting surface $\widetilde{M}$ an elliptic fibration. If we introduce a homogeneous coordinate z on the bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ over the exceptional $\mathbb{P}^1$, this surface is given explicitly by

$$-y^2 + x^3 + fz^4v^4x + gz^6v^6 + z^6u\big(\alpha_5(u,v) + \ldots + \alpha_0(u,v)xy\big) = 0 \; . \tag{6}$$

The point $u = v = 0$ also corresponds to the base point of the system of cubics in $\mathbb{P}^2$ which pass through the eight specified points in the definition of $M \cong \mathrm{dP}_1$; in this way we see that $\widetilde{M}$ is in fact a rational elliptic surface.

In the context of F-theory heterotic duality, the fibre over $u = 0$ is identified with the heterotic elliptic curve, and the unfolding of the singularity at $v = 0$ corresponds to the heterotic bundle.

To obtain the F-theory picture dual to a full $E_8 \times E_8$ model, we need two such surfaces $\widetilde{M}_1$ and $\widetilde{M}_2$; by gluing them together along the common elliptic curve $\mathcal{E}$, we get a degenerate $K3$ surface. There is an elementary way to see that this is indeed a $K3$:

An elliptic $K3$ has base $\mathbb{P}^1$, and generically 24 singular fibres. Let (t_0, t_1, t_2) be homogeneous coordinates on $\mathbb{P}^2$, and consider the family of quadric curves given by

$$C_\lambda : \; t_0t_1 - \lambda^2\, t_2^2 = 0 \; , \; \lambda \in \mathbb{C} \; .$$

Any $\lambda \neq 0$ defines a smooth rational curve, however for $\lambda = 0$, this degenerates to two rational curves intersecting transversely at a point. If we define a family of $K3$ surfaces elliptically fibred over this, then at $\lambda = 0$ they will degenerate to a pair of elliptic surfaces glued along the common fibre at $t_0 = t_1 = 0$. The anti-canonical class of the curve is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$, so the discriminant of the Weierstrass model is a section of $\mathcal{O}_{\mathbb{P}^2}(12)$. Therefore on the degenerate fibre there are twelve singular elliptic fibres over each $\mathbb{P}^1$ component of the base, making each component of the degenerate fibre a rational elliptic surface.

3 Duality in lower dimensions

To obtain the duality in lower dimensions, we fibre the above constructions over some manifold. Since we are most interested in compactifications to four dimensions, we will take this to be a surface S; the results are easily applied to other dimensions.

3.1 The heterotic picture

On the heterotic side, we now have an elliptically-fibred Calabi–Yau threefold Z, with projection $\pi_Z : Z \to S$; the results of Section 1 tell us that we must take the homogeneous coordinates on the fibre, (X, Y, Z), to transform like sections of $\mathcal{O}_S(-2K_S) \oplus \mathcal{O}_S(-3K_S) \oplus \mathcal{O}_S$, and therefore the parameters f, g to be sections of $\mathcal{O}_S(-4K_S)$ and $\mathcal{O}_S(-6K_S)$.

Let us suppose that our heterotic bundle restricts to a semi-stable bundle on the general fibre of Z. Then on each fibre it has the description given previously, and as we move over the base S, (3) and (4) together give us an n-fold cover of S, called the spectral cover.

Since x and y now transform non-trivially as we move on S, the spectral cover equation (4) only makes sense if the parameter a_m is a section of $\mathcal{O}_S(mK_S + \eta)$, where η is some sufficiently ample divisor class on S (which doesn't depend on m, obviously).

3.1.1 Constructing bundles from a spectral cover

The spectral cover C does not uniquely determine a bundle on Z. Instead, we recover our bundle V as $\pi_*(\mathcal{L} \otimes \mathcal{P})$, where $\mathcal{P}$ is the 'Poincaré line bundle', π is the projection from $C \times Z$ to Z, and $\mathcal{L}$ is some line bundle on C satisfying

$$c_1(\mathcal{L}) = -\frac{1}{2}\big(c_1(C) - \pi_C^* c_1(S)\big) + \gamma \ ,$$

where γ is any class on C satisfying $\pi_{C*}\gamma = 0$. This ensures that the resulting bundle V satisfies $c_1(V) = 0$.

The only general construction for the class γ is

$$\gamma = \lambda\Big(n\sigma - \pi_C^*\big(\eta - n\, c_1(B)\big)\Big) \ , \ \lambda \in \mathbb{Z} \ .$$

With this choice of γ, the third Chern class of V is

$$c_3(V) = 2\lambda\eta \cdot_B (\eta - n\, c_1(B)) \ .$$

3.1.2 Localisation of cohomology

The massless spectrum of a heterotic model corresponds to cohomology groups of V and associated bundles. We can use the fibration structure of Z to assist with such calculations; in particular, we have the Leray spectral sequence:

$$E_2^{p,q} = H^p(S, R^q\pi_{Z*}V) \ \longrightarrow \ H^{p+q}(Z, V) \ ,$$

where the higher direct image $R^q\pi_{Z*}V$ is the sheaf associated to the pre-sheaf on S given by $S \supset U \mapsto H^q(\pi_Z^{-1}(U), V)$.

Observe that on a generic fibre $\mathcal{E}_s$, V decomposes as a direct sum of non-trivial degree-zero line bundles, and as such $H^*(\mathcal{E}_s, V\big|_{\mathcal{E}_S}) = 0$. The cohomology of V will therefore be localised on the locus over which V contains a trivial piece. This corresponds to one of the points Q_i becoming coincident with p_0, which is at infinity in the affine coordinates x, y. Since the n points Q_i are determined by a degree n polynomial of the form $\ldots + a_n x^n = 0$, moving one of them to infinity corresponds to $a_n \to 0$.

So the cohomology of V is localised over the divisor on S given by $a_n = 0$, in divisor class $n\, K_S + \eta$.

3.2 The F-theory picture

On the F-theory side, we begin by considering a family of surfaces fibred over S, each given by an equation of the form of (5), making up a fourfold Y. Since we must still identify the heterotic fibre $\mathcal{E}$ with the curve given by $u = 0$ over each point, we learn that (x, y, v) transform as sections of $\mathcal{O}_S(-2K_S), \mathcal{O}_S(-3K_S)$, and $\mathcal{O}_S$ respectively. The transformation properties of u, on the other hand, are not fixed, but correlated with the transformation properties of the coefficients in the functions α_k. We will demonstrate this with an example:

The case of unbroken $SU(5)$

Let $u \sim \mathcal{O}_S(\hat{\eta})$ for some divisor class $\hat{\eta}$. Suppose we want to generate an A_4 singularity at $v = 0$. Then by Tate's algorithm, we must take the equation for Y to be (setting $z \to 1$, which we can do in the relevant patch)

$$p_{\mathcal{E}}(v, x, y) + u(a_0 v^5 + a_2 v^3 x + a_3 v^2 y + a_4 v x^2 + a_5 x y) = 0 \ .$$

In order for this to be well-defined, since the terms in $p_{\mathcal{E}}$ are sections of $\mathcal{O}_S(-6K_S)$, we must take the a_m to be sections of the following bundles:

$$a_m \sim \mathcal{O}_S\big((m-6)K_S - \hat{\eta}\big) \ .$$

Therefore if we identify $\hat{\eta} \sim -6K_S - \eta$, where η is the divisor class entering the spectral cover data on the heterotic side, we get an exact match between the two moduli spaces.

3.2.1 Moving away from the stable degeneration

Each rational elliptic surface fibre has a $\mathbb{P}^1$ base parametrised by $[u : v]$, and we have seen that $v/u \sim 6K_S + \eta$. This is the normal coordinate to the divisor on which the gauge group lives. What happens when we deform away from the stable degeneration limit?

Consider again the family of rational curves given by

$$C_\lambda : \ t_0 t_1 - \lambda^2 t_2^2 = 0 \ , \ \lambda \in \mathbb{C} \ .$$

For $\lambda \neq 0$, the explicit isomorphism with $\mathbb{P}^1$ is $(t_0, t_1, t_2) = (\lambda s_0^2, \lambda s_1^2, s_0 s_1)$. For any λ, we therefore have two distinguished divisors:

$$\begin{aligned} D_0 : \ t_0 = t_2 = 0 \quad &\longleftrightarrow \quad s_0 = 0 \ , \\ D_1 : \ t_1 = t_2 = 0 \quad &\longleftrightarrow \quad s_1 = 0 \ , \end{aligned}$$

with normal coordinates $s_0/s_1 = \lambda t_2/t_1$ and $s_1/s_0 = \lambda t_2/t_0$ respectively.

The singular curve C_0 has two components, each isomorphic to $\mathbb{P}^1$. The first is parametrised by $[t_1 : t_2]$ and contains the divisor D_0 at $t_2 = 0$, while the second is parametrised by $[t_0 : t_2]$ and contains the divisor D_1 at $t_2 = 0$. The normal coordinate to D_0 is therefore t_2/t_1, and that to D_1 is t_2/t_0.

Moving away from the stable degeneration point corresponds to turning on a non-zero λ, and the heterotic manifold at $t_0 = t_1 = 0$ is lost, leaving only the two divisors D_0 and D_1, over which the two gauge groups live. We saw above that this does not change the local topology near each divisor, which is as one would expect. However, we now have an identification between the two normal bundles: $D_0 \cong D_1$, but the normal bundles are related by $\mathcal{N}_{D_1|B} \cong \mathcal{N}_{D_0|B}^{-1}$, as the two divisors now intersect each fibre at opposite poles of a common $\mathbb{P}^1$. In terms of the classes η, we therefore have

$$\begin{aligned} 6K_S + \eta_1 &= -6K_S - \eta_0 \\ \Rightarrow \ \eta_0 + \eta_1 &= -12K_S \ . \end{aligned}$$

<u>Example:</u>

We can translate the analysis of Morrison and Vafa into this language. The heterotic theory is defined on an elliptic $K3$, so the base B on the F-theory side is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, i.e., a Hirzebruch surface. These are distinguished by a parameter n:

$$B = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^n}(n)\big) \ .$$

The two divisors D_0 and D_1 live at the poles of the fibration, and therefore have normal bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$ and $\mathcal{O}_{\mathbb{P}^1}(n)$ respectively. Since $K_{\mathbb{P}^1} = -2$, this lets us identify the classes $\eta_0 = 12 - n$ and $\eta_1 = 12 + n$. For any n, there is therefore ample freedom in the bundle construction to completely break the second E_8. However for large enough n, we will leave some subgroup of the first E_8 unbroken.
For example, if $n = 5$, then $mK_{\mathbb{P}^1} + \eta_0 = 7 - 2m$, and for $m > 3$ this has no sections. Therefore we can turn on at most an $SU(3)$ bundle, breaking E_8 to E_6.