SU(5) grand unification in an extra dimension

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Abstract

This thesis describes work towards building an $SU(5)$ grand unified field theory in a 4+1-dimensional spacetime. After reviewing important work on field theory and extra dimensions, we study some general features of the type of model we wish to develop. Translation invariance in the extra dimension is broken by the formation of a scalar domain wall, which facilitates the localisation of other fields, thus yielding an effective 3+1-dimensional theory at low energies. Having understood the general issues, we present a specific model based on gauged $SU(5)$ symmetry, and analyse some of its properties. Finally we point out where more work needs to be done.

Declaration

This is to certify that

(i) the thesis comprises only my original work towards the Masters except where indicated in the Preface,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is less than 30,000 words in length, exclusive of tables, bibliographies and appendices.

Preface

Chapters 2-4 comprise an original review by the author of well-known material. Chapter 5 is original work by the author, except where acknowledged in the text. In particular, section 5.1 represents independent work, even though the main results were also obtained in Ref. [1]. Chapter 6 is based on Ref. [2], which is a collaboration between the author and D.P. George. The content and discussion throughout chapter 6 was decided together, the calculations preceding section 6.1 were originally performed by the present author, and the calculations/numerical work in section 6.1 were performed by George. The prose in chapter 6 was written by the present author. Chapter 7 is entirely original work by the author, aided by discussions with R.R. Volkas. The original idea and the equations of motion are presented in [3], but all results are new. Chapter 8 is based on Ref. [4], a collaboration between the author, D.P. George and R.R. Volkas. Section 8.1, 8.2, 8.4 are essentially unchanged from the paper and contain substantial contributions from George and Volkas, but the rest of chapter 8 is further work done by the author alone, aided by discussions with the other collaborators. Appendix A is original work. Appendices B and D are reviews of standard material. The calculations in Appendix C were performed independently by the author, but represent very well-known results.

Contents

Chapter 1

Introduction

Our current understanding of the fundamental interactions in the universe (excluding gravity) is encapsulated in the Standard Model of particle physics, a quantum field theory with gauge symmetry $SU(3) \times SU(2) \times U(1)$. With a relatively small number of parameters, the theory has been amazingly successful at explaining a wide variety of phenomena at the sub-atomic level. It does however have a number of features that are aesthetically displeasing – among other things it suffers from the 'hierarchy problem', and provides no explanation for the ratio of the various fermion masses or the existence of three generations of fermions. Various theoretical extensions of the Standard Model have been proposed to alleviate one or more of these problems, but the alternatives can only be distinguished by experiments at higher energies.

The Large Hadron Collider (LHC) is a next-generation particle accelerator being built at CERN, and due to begin collecting data in mid-2008. Once fully operational, the LHC will collide particles at centre-of-mass energies up to 14 TeV - almost 100 times higher than any experiment before it. This is expected to reveal phenomena which have never before been observed, and take us one step closer to understanding the fundamental structure of our universe.

Among the most popular ideas for which evidence may be found at the LHC are the Standard Model Higgs boson, supersymmetry, and the main focus of this project, extra spatial dimensions.

The possibility of extra dimensions has become widely accepted in recent decades, largely because of the meteoric rise of string theory, which is only internally consistent in the presence of extra dimensions [5].

As well as the continuous stream of work on string theory, there has been a considerable amount of work done on field-theoretical models of extra dimensions. The first obvious objection to this approach is that most interacting field theories (including gauge theories) are non-renormalisable in more than four spacetime dimensions. However, the modern point of view is that quantum field theory cannot be a fundamental description of the universe; for example, it seems unable to incorporate gravitation. Therefore every field theory is envisaged to come with a cut-off – an energy scale above which a new description of physics is necessary. The quantum corrections in a non-renormalisable theory are dependent on this cut-off, but they are finite. Therefore if there is a sensible way to determine the cut-off, a non-renormalisable theory is predictive.

In the form of the Standard Model of particle physics, quantum field theory has been extraordinarily successful in describing physics up to energies of order 100 GeV. With no pointers yet from experiment to the nature of the ultimate theory of everything, it thus seems sensible to rely on field theory to investigate what we might find at the LHC.

The ultimate goal of this project is to develop a field theory in a 5D spacetime which nonetheless reproduces the 4D Standard Model of particle physics at low energies. It seems arbitrary to single out certain spatial dimensions to have different topology, so we will try to develop a model in five non-compact dimensions i.e. a spacetime homeomorphic to $\mathbb{R}^5$.

In part I we review the most relevant aspects of extra-dimensional physics, field theory, and grand unification. Chapter 2 details the most popular recent models with compact extra dimensions, while chapter 3 introduces theories with non-compact extra dimensions. Chapter 4 outlines grand unification with the gauge group $SU(5)$, as well as the concepts of running coupling constants and confinement in field theory, and finally charge conjugation of fermion fields in $4+1$ dimensions. Part II consists of a study of the general aspects of physics with a warped extra dimension. Chapter 5 is concerned with analysing if and how ordinary 4D gravity is reproduced in this scenario, and in chapter 6 we introduce fermion and scalar fields to study whether or not they can be appropriately localised in such models. In part III we begin to develop a (hopefully) realistic $SU(5)$ grand unified model in a warped extra dimension. In chapter 7 we obtain a consistent solution of the Klein-Gordon and Einstein equations which yields appropriate warping of the extra dimension. In chapter 8 we introduce the matter fields of the model, study their localisation, and perform some preliminary analysis of the model. We also point out some of the questions which still need to be addressed. Finally, the appendices contain some calculations and techniques used in the main body of the thesis.

1.0.1 Notation and Conventions

Repeated indices are always summed over, unless specifically stated otherwise.

The sign convention for the spacetime metric is $(+, -, \cdots, -)$ throughout. Upper case Latin letters $M, N, \ldots$ represent 5D spacetime indices, while lower case Greek letters $\mu, \nu, \ldots$ are used for 4D spacetime indices. In the few cases that it is necessary, upper case Latin letters from the start of the alphabet $(A, B, \ldots)$ are used for 5D flat space or internal Lorentz indices (see appendix B).

An upper case D denotes a covariant derivative. This could be a gravitational covariant derivative, a gauge-covariant derivative, or both, which will be clear from context.

The conventions for the Riemann curvature tensor are such that $(D_\kappa D_\tau - D_\tau D_\kappa)X^\mu = R^\mu{}_{\nu\kappa\tau}X^\nu$, where D is the gravitational covariant derivative.

For Feynman diagram calculations there will always be an implicit $+i\epsilon$ in the denominator of propagators; to avoid clutter we will not display it.

For convenience we identify all relevant Lie groups with their fundamental representations as matrices. The generators of a Lie group will be denoted by $\{t^a\}$, so that a generic element of the group can be written as[1] $\exp(-i\theta^a t^a)$ with $\theta^a \in \mathbb{R}$. For a unitary matrix group, the generators will be Hermitian matrices.

[1]Note that this is only strictly possible because we only consider connected groups herein.

The normalisation will always be $\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$. For some other representation $\mathbf{r}$, we denote the representation of the generators by $\{t^a_{\mathbf{r}}\}$, and define $C(\mathbf{r})$ by $\mathrm{Tr}[t^a_{\mathbf{r}} t^b_{\mathbf{r}}] = C(\mathbf{r})\delta^{ab}$. The quadratic Casimir operator is defined by $C_2(\mathbf{r}) = t^a_{\mathbf{r}} t^a_{\mathbf{r}}$. The covariant derivative for a field in a representation $\mathbf{r}$ is $D_\mu = \partial_\mu - igA^a_\mu t^a_{\mathbf{r}}$.

Part I

Background material

Chapter 2

Compact Extra Dimensions

The idea that our universe may have more dimensions than we observe has been around for over 80 years. The earliest model utilising this possibility is now called the Kaluza-Klein model, and attempted to unify Maxwell's theory of electrodynamics with Einstein's theory of General Relativity [6, 7]. Although this original attempt has since been abandoned, the idea of unobserved spacetime dimensions has reappeared in various guises, and much recent theoretical physics literature is concerned with extra dimensions. In this project we will be exploring the logical possibility that, at low energies, our universe is described by a field theory in a higher-dimensional spacetime.

Of course, if these theories are to accurately describe experiment, it is necessary to explain why we have only ever observed 3+1 dimensions. Attempts to do so fall into three main categories:

- Theories with 'universal' compact extra dimensions [8]. Here all fields propagate in all spacetime dimensions, but the extra dimensions are too small to have been observed in any experiments (in a sense which will be made precise later).

- Theories with compact extra dimensions in which only gravity propagates [9]. The Standard Model particles are trapped on a 3+1-dimensional brane. The extra dimensions in these models can be reasonably large, due to the fact that gravity has not been measured accurately at small distances.

- Theories with non-compact extra dimensions [10]. Here the extra dimensions are infinite in size, but hidden from us because all the particles we observe are localised to a 3+1-dimensional subset of the full spacetime.

Although the theory presented later in this thesis belongs to the last category, we will initially consider the others. This will introduce in a simpler context some of the general types of results of extra-dimensional physics which we will encounter later, as well as being important in its own right to understand much of the recent literature.

2.1 General considerations

Suppose the topology of the universe is $\mathbb{R}^4 \times N^n$, where N^n is some compact n-manifold representing extra spatial dimensions. Geometrically, we suppose that the size of N^n is characterised by some small length scale R. We will now demonstrate that at energies smaller than approximately $1/R$, the extra dimensions will be hidden from physics experiments. Take for example a free scalar field Φ, described by the action[1]

$$\mathcal{S} = \int d^4x \int_{N^n} \frac{1}{2}(\partial_M \Phi \partial^M \Phi - m^2 \Phi^2). \tag{2.1}$$

where the coordinates x label just the 4 familiar dimensions. The equation of motion following from this action is the $4+n$-dimensional **Klein Gordon equation**:

$$\Box_{4+n}\Phi + m^2\Phi = 0 \tag{2.2}$$

where $\Box_{4+n}$ is the d'Alembertian operator on the whole spacetime. We can split this operator up into the usual 4D d'Alembertian and the Laplacian for N^n, so that the equation reads

$$\Box_4\Phi - \nabla_N^2\Phi + m^2\Phi = 0. \tag{2.3}$$

[1]We have written the integral over N^n as $\int_{N^n}$ to avoid complications arising from the possibly non-trivial geometry of N^n.

We can introduce coordinates y on N^n, and expand the y-dependence of Φ in a generalised Fourier series:

$$\Phi(x,y) = \sum_j f_j(y)\phi_j(x) \tag{2.4}$$

where f_j is some complete set of functions on N^n. We can find the f_j by demanding that each ϕ_j satisfy the 4-dimensional Minkowski space Klein-Gordon equation, with some as-yet undetermined mass:

$$\Box_4\phi_j + m_j^2\phi_j = 0 \tag{2.5}$$

This leaves us with the following equation for each value of the index j:

$$\nabla_N^2 f_j(y) = (m^2 - m_j^2)f_j(y) \tag{2.6}$$

So the f_j are the eigenfunctions of the Laplacian on N^n. If we label the eigenvalues of ∇_N^2 as[2] $-\lambda_j$, the above equation allows us to find the masses m_j:

$$m_j^2 = m^2 + \lambda_j \tag{2.7}$$

Note that we always get a zero mode characterised by $\lambda_0 = 0$ and thus $m_0 = m$. This corresponds to a function that is constant on N^n. Using these results, and the fact that the f_j, being eigenfunctions of a Laplacian, form an orthonormal set, we can normalise them appropriately and perform the integral over N^n in Eq. (2.1) to obtain

$$\mathcal{S} = \int d^4x \frac{1}{2}\sum_j (\partial_\mu\phi_j\partial^\mu\phi_j - (m^2+\lambda_j)\phi_j^2). \tag{2.8}$$

We now observe a characteristic feature of these theories. Although we have retained all of the degrees of freedom, we have re-written the action as that of a purely 3+1-dimensional theory. The existence of the extra dimensions is encoded in the fact that we now have an infinite "tower" of 4D particles with masses $\sqrt{m^2+\lambda_j}$. These are often referred to as 'Kaluza-Klein' (KK) modes or excitations. The processing of re-writing the theory this way is called dimensional reduction.

[2]The Laplacian operator on any Riemannian manifold is negative semi-definite, so all λ_j thus defined are non-negative.

We can now understand why the extra dimensions may not be detectable at low energies. By dimensional analysis, we can argue that the smallest eigenvalues of ∇_N^2 must be of order $1/R^2$. We can see from Eqs. (2.7) and (2.8) that detecting any mode of Φ other than the zero mode corresponds to producing a particle of some mass greater than about $1/R$. If R is very small, this can be well beyond current experimental capabilities.

2.2 The Arkani-Hamed, Dimopoulos and Dvali model

The ADD model, as it has become known, was first presented in Ref. [9] as a way to solve the hierarchy problem. The idea is to have compact extra dimensions, but confine all fields other than gravity to a 3+1-dimensional slice, or 'brane', so that they are not directly affected. This allows the extra dimensions to be quite large, because although particle physics experiments have been performed down to distances much smaller than the size of an atomic nucleus, gravity has only been measured on scales down to approximately .1mm.

The mechanism by which this model explains the hierarchy problem is very simple. Newton's gravitational constant G_N can be written in terms of the Planck mass[3] M_{Pl} as $G_N = 1/M_{Pl}^2$, so that the gravitational acceleration at a distance r from a point mass m is given by

$$g(r) = \frac{1}{M_{Pl}^2}\frac{m}{r^2}. \tag{2.9}$$

It is well known that the $1/r^2$ dependence can be understood in terms of gravitational flux spreading out isotropically in 3-dimensional space. Correspondingly, if the fundamental gravitational mass scale of the universe is M, then in $4+n$ dimensions the equivalent of Eq. (2.9) is

$$g(r) = \frac{1}{M^{2+n}}\frac{m}{r^{2+n}}. \tag{2.10}$$

[3] Actually, M_{Pl} used here is more accurately called the "modified Planck mass". It differs from the conventional value by an inconsequential numerical factor.

But now suppose that the extra n dimensions are compact, with characteristic size R. Then at distances $r > R$, these dimensions become 'saturated' with gravitational flux, and the dependence will revert to $1/r^2$. For distances $r > R$, we therefore expect

$$g(r) = \frac{1}{M^{2+n} R^n} \frac{m}{r^2}. \tag{2.11}$$

The result is that what we identify as the Planck mass in 4D is related to the scale M via

$$M_{Pl}^2 = M^2 (M^n R^n). \tag{2.12}$$

So as long as R is larger than $1/M$, and n is large enough, it is possible to obtain the observed value of the Planck mass even with M as low as the electroweak scale. This solves the hierarchy problem in its usual guise, but does introduce a new hierarchy – that between R and $1/M$.

2.3 The type-1 Randall-Sundrum model

The ADD model solves the hierarchy problem by bringing the fundamental gravitational mass scale down to approximately the electroweak scale. In Ref. [11], Randall and Sundrum took the opposite approach, and showed that a small effective 4D electroweak scale can be obtained in a theory where all mass parameters are of order the Planck mass. This is achieved by replacing the flat extra dimensions of the ADD model with a "warped" solution of Einstein's equation. We will refer to this theory as RS1.

The model has just a single extra dimension which, instead of being circular, is an interval described by the coordinate ϕ, with $0 \leq \phi \leq \pi$. In order to induce the "warping" of the extra dimension, two branes with non-zero energy density are introduced, one at $\phi = 0$ and one at $\phi = \pi$. All fields except gravity are forced to reside on the brane at $\phi = \pi$.

The following solution is found for the metric:

$$ds^2 = e^{-2kr_c|\phi|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \tag{2.13}$$

where $y = r_c \phi$. The interesting feature of this metric is that the resulting geometry is non-factorisable ie. the spacetime cannot be written as the product of two (pseudo-)Riemannian manifolds. Given this gravitational background, we can ask how physics on the brane at $y = \pi r_c$ will be affected. Consider a Higgs field H, with action (remembering that H is *a priori* confined to the brane at $y = \pi r_c$),

$$\mathcal{S}_H = \int_{y=\pi r_c} d^4x \sqrt{g} \left\{ g^{\mu\nu} \partial_\mu H^\dagger \partial_\nu H - \lambda(|H|^2 - v_0^2)^2 \right\} . \tag{2.14}$$

We can now substitute in g from the 4D part of the solution in Eq. (2.13) to get

$$\mathcal{S}_H = \int d^4x e^{-4kr_c\pi} \left\{ e^{2kr_c\pi} \eta^{\mu\nu} \partial_\mu H^\dagger \partial_\nu H - \lambda(|H|^2 - v_0^2)^2 \right\} . \tag{2.15}$$

Notice that we now have to rescale the field H in order to obtain a correctly normalised kinetic term. With $H \to e^{kr_c\pi} H$, the action becomes

$$\mathcal{S}_H = \int d^4x \left\{ \eta^{\mu\nu} \partial_\mu H^\dagger \partial_\nu H - \lambda(|H|^2 - e^{-2kr_c\pi} v_0^2)^2 \right\} . \tag{2.16}$$

So the scale of symmetry breaking on the brane is not v_0 as would be naïvely expected, but $e^{-kr_c\pi} v_0$. This means that the fundamental parameter v_0 can be of order the Planck mass, but lead to electroweak scale symmetry breaking on the brane, as long as $kr_c \simeq 50$. Thanks to the exponential factor, the usual hierarchy of order 10^{17} between fundamental parameters is reduced to about 50. The hierarchy problem is gone!

Discussion of how 4D gravity is reproduced in this scenario will be postponed until we present the type-2 Randall-Sundrum model later.

Chapter 3

Non-compact Extra Dimensions

As has already been mentioned, the idea that the universe may have compact extra dimensions goes back at least to the work of Kaluza and Klein in the 1920s. More recently it has been realised that it is actually possible for the universe to have *non*-compact extra dimensions without contradicting the apparent 3+1-dimensionality we observe at low energies.

3.1 Rubakov and Shaposhnikov's domain wall mechanism

In Ref. [12], Rubakov and Shaposhnikov presented a mechanism for localising chiral fermions to a 3+1-dimensional slice of a 4+1-dimensional spacetime. The model starts with a single real scalar field η, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_A \eta \partial^A \eta + \frac{1}{2} m^2 \eta^2 - \frac{\lambda}{4}\eta^4. \tag{3.1}$$

This theory admits a classical solution $\eta^{cl}(y)$ depending on only one of the spatial coordinates, which we choose to be the extra dimension:

$$\eta^{cl}(y) = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{my}{\sqrt{2}}\right) \tag{3.2}$$

This type of configuration is referred to as a domain wall; it interpolates between the two global minima of the potential.

We can now introduce a fermion field Ψ, which couples to the scalar background via the Yukawa Lagrangian

$$\mathcal{L}_\Psi = i\overline{\Psi}\Gamma^A\partial_A\Psi - h\overline{\Psi}\Psi\eta, \tag{3.3}$$

where the 5D Dirac matrices are given in terms of the standard 4D matrices by

$$\Gamma^\mu = \gamma^\mu, \quad \mu = 0, \ldots, 3 \quad \Gamma^5 = -i\gamma^5. \tag{3.4}$$

In the background of Eq. (3.2), the Dirac equation admits a separable solution $\Psi(x, y) = f(y)\psi(x)$, where ψ is a left-handed[1] massless 4D spinor, and f is given by

$$f(y) \propto \exp\left(-h\int^y ds\,\eta^{cl}(s)\right). \tag{3.5}$$

This solution decays exponentially away from the centre of the domain wall. We say that the left-handed zero mode is localised to the wall. There are also massive solutions, but the lightest of these has mass of order $hm/\sqrt{\lambda}$. Therefore at low energies, these modes will not be produced, and the theory can be described by an effective field theory of massless *chiral* fermions in 4D. This is a very attractive mechanism for model-building purposes.

It was also demonstrated that the field Φ itself has a zero mode localised to the domain wall, as well as a tower of massive modes, just like the fermions.

It is also possible to couple other 5D scalar fields to Φ, and thus localise these scalars to the wall as well. A detailed analysis of fermions and scalars coupled to a domain wall is performed in Ref. [13].

Of course, this model says nothing about gravity. We would expect that the 4+1-dimensional nature of the universe would be revealed in a $1/r^3$ form for Newton's law, thus contradicting observations. This problem was solved by Randall and Sundrum in 1999.

[1] 4D chirality enters because of the inclusion of γ^5 in the 5D Clifford algebra, as per Eq. (3.4). There is no normalisable right-handed solution. If the sign of h is changed, a right-handed mode is localised instead.

3.2 The type-2 Randall-Sundrum model

In Ref. [10], Randall and Sundrum presented a model with 4 non-compact spatial dimensions which nonetheless reproduces familiar $1/r^2$ gravity at large distances.

The starting point is in fact the type-1 Randall-Sundrum model described in section 2.3. We take the same metric solution given in Eq. (2.13), but this time assume that all non-gravitational fields reside on the brane at $\phi = 0$.

First we derive the relationship between the familiar 4D Planck scale M_{Pl} and the fundamental scale M. To do this we perform dimensional reduction on the Einstein-Hilbert action in the background of Eq. (2.13). Explicitly, suppose there are 4D gravitational fluctuations, such that the Minkowski metric is replaced by some other metric $\overline{g}_{\mu\nu}(x)$. If $\overline{R}$ denotes the Ricci scalar determined by $\overline{g}_{\mu\nu}$, then the dimensional reduction of the Einstein-Hilbert action is given by

$$-\int d^4x M_{Pl}^2 \sqrt{\overline{g}}\,\overline{R} = -\int d^4x \int_0^{\pi r_c} dy\, 2M^3 e^{-2k|y|} \sqrt{\overline{g}}\,\overline{R} \tag{3.6}$$

$$\Rightarrow M_{Pl}^2 = 2M^3 \int_0^{\pi r_c} dy\, e^{-2k|y|} \tag{3.7}$$

$$= \frac{M^3}{k}[1 - e^{-2kr_c\pi}]. \tag{3.8}$$

It is clear from this expression that a sensible limit can be obtained by taking $r_c \to \infty$, thus making the extra dimension non-compact. Due to the fact that the solution is symmetrical under $y \to -y$, we can also double the spacetime, so that $-\infty < y < \infty$, and get a non-compact spacetime without boundary. The non-gravitational fields still reside on the brane at $y = 0$.

We have not yet demonstrated that gravity behaves as 3+1-dimensional in this scenario. To do so, we need to consider fluctuations of the 4D components of the metric, to which the matter fields couple. Specifically, we define $h_{\mu\nu}$ by $g_{\mu\nu} = e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(x, y)$. To determine the 4D graviton spectrum, we expand h in a generalised Fourier series in y just as we did earlier for a Klein-Gordon field. For gravitons though, we have some gauge freedom, coming from the invariance of general relativity under coordinate changes. We choose the

4D graviton to satisfy the conditions of the transverse-traceless gauge, so that $\partial^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$. Working in this gauge, let $\psi^m(y)$ be the extra-dimensional profile of the mode of mass m. Then it satisfies the equation

$$\left[-\frac{m^2}{2}e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\right]\psi^m(y) = 0. \tag{3.9}$$

This can be put into the form of a Schrödinger equation via the following change of variables:

$$z \equiv \frac{1}{k}sgn(y)\left(e^{k|y|} - 1\right) \quad \hat{\psi}^m \equiv \psi^m e^{\frac{k|y|}{2}}. \tag{3.10}$$

Substituting these definitions into Eq. (3.9) yields

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right]\hat{\psi}^m(z) = m^2\hat{\psi}^m(z), \tag{3.11}$$

where

$$V(z) = \frac{15k^2}{8(k|z|+1)^2} - \frac{3k}{2}\delta(z). \tag{3.12}$$

This can now be analysed using the well-known techniques of 1D quantum mechanics. There is a solution corresponding to $m = 0$, which we identify with the usual graviton. For later use we record that its extra-dimensional profile is proportional to the warp factor: $\psi^0(y) \propto e^{-2k|y|}$.

Because $V(z) \to 0$ as $|z| \to \infty$, there is also a continuum of modes starting at $m = 0$. This is potentially disastrous, as the extra dimensions will not be hidden by a mass gap. However, the continuum modes are explicitly derived in Ref. [10] and it is demonstrated that their extra-dimensional profiles are suppressed by a factor of approximately m/k at the position of the brane, where m is the mass of the state. This means that the light continuum modes couple much more weakly to the matter on the brane than does the zero mode. The resulting generalisation of the Newtonian potential between two masses on the brane is

$$V(r) = G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{r^2k^2}\right). \tag{3.13}$$

Therefore at distances large compared to $1/k$, the usual Newtonian potential is approximately recovered. This explains why the presence of the extra dimension does not contradict previous tests of Newton's law. It is often said that the warped metric 'localises gravity on the brane'.

3.2.1 Smooth versions of warped gravity

From a field theory point of view, the infinitely thin 'brane' which produces warped gravity in the RS2 model is a strange object. It would be more desirable to find a way to generate a brane-like object from another 5D field. Many models have been described in which scalar field domain walls induce Randall-Sundrum-like gravity [14, 4, 15, 16, 17, 18, 19, 20]. This works because the energy density of a domain wall is concentrated around the centre of the wall, and thus approximates the brane in Ref. [10].

We will see below that warped gravity fails to localise fermions to the brane. In smooth versions of the Randall-Sundrum model however, it is possible for the domain wall to again localise massless chiral fermions as per the idea of Rubakov and Shaposhnikov. We will see this in detail in chapter 6.

3.3 Localisation by warped gravity

In the case of compact extra dimensions, it is the small size of the extra dimensions that explains our inability to detect them so far. In the case of non-compact extra dimensions, however, we need to find other mechanisms to shield low-energy physics from the extra dimension. In Ref. [10], fields other than gravity are *a priori* confined to an infinitely thin 'brane'; this is too ad hoc from a field theory point of view.

It is possible that the warped gravity of the Randall-Sundrum model may localise fields other than gravity. This idea was thoroughly explored by Bajc and Gabadadze in Ref. [21]. Here we review their findings. We note that although we use the original solution with an infinitely thin brane, the results here only depend on the asymptotic form of the metric, and thus apply also to smooth versions of the RS2 scenario.

All that is required is to write down the equations of motion for various spin fields in the background of Eq. (2.13).

Spin-0 fields

Suppose we have a 5D scalar field Φ. We expand Φ as usual,

$$\Phi(x,y) = \sum_m \chi^m(y)\phi^m(x), \tag{3.14}$$

where ϕ^m is a 4D Klein-Gordon field of mass m. If we introduce a new function $u^m(y) = e^{-2k|y|}\chi^m(y)$, the resulting equation is

$$\left[-\frac{m^2}{2}e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\right] u^m(y) = 0. \tag{3.15}$$

This is exactly Eq. (3.9), which determines the spectrum of gravitons. Therefore we know that there is a solution for $m = 0$ that is proportional to the warp factor: $u^0(y) = ce^{-2k|y|}$. This means that the zero mode is in fact constant along the extra dimension: $\chi^0(y) = c$. It seems then that gravity does not localise a zero mode scalar field, for this function is not peaked near $y = 0$. However, the presence of the warp factor in the metric means that the zero mode is in fact properly normalisable, which we can see from examining the action:

$$\int d^4x \int dy\sqrt{g}\left[\frac{1}{2}g^{MN}\partial_M\Phi\partial_N\Phi\right] \supset \int dye^{-2k|y|}c^2 \int d^4x\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0. \tag{3.16}$$

We conclude that Randall-Sundrum gravity is capable of localising a massless scalar field to the brane.

Spin-1 fields

Let A_M be a 5D vector field. We work in the gauge where $A_5 = 0$. Then its equation of motion admits a zero mode solution $v^0(y)a_\mu(x)$, where $v^0(y)$ is constant, just as in the scalar field case. The action for a vector field has a different form though, and it turns out that this solution is not normalisable. We can demonstrate this explicitly by defining $F_{MN} = \partial_M A_N - \partial_N A_M$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Then the kinetic part of the vector field action (even for non-Abelian fields) is

$$\begin{aligned}\mathcal{S}_A &= -\frac{1}{4}\int d^4x \int dy\sqrt{g}\, g^{MN}g^{PQ}F_{MP}F_{NQ} \\ &\supset -\frac{1}{4}\int d^4x\eta^{\mu\nu}\eta^{\kappa\tau}f_{\mu\kappa}f_{\nu\tau}\int dy\,(v^0(y))^2,\end{aligned} \tag{3.17}$$

which is clearly infinite for $v^0(y)$ constant. So warped gravity in five dimensions cannot localise a massless vector field.

Interestingly, there *is* a way to localise gauge bosons with warped gravity alone, but it involves introducing more than one extra space dimension. The simplest example is given in Ref. [22], where it is assumed that as well as a single non-compact extra dimension, we also have n circular extra dimensions, of small radius R. The metric is

$$ds^2 = e^{-k|y|}(\eta_{\mu\nu}dx^\mu dx^\nu - \sum_{i=1}^{n} R^2 d\theta^{i2}) - dy^2 \tag{3.18}$$

The determinant of the metric now contains extra factors of $e^{k|y|}$, so that the analogue of Eq. (3.17) is

$$\mathcal{S} \supset -\frac{1}{4}\int d^4x\eta^{\mu\nu}\eta^{\kappa\tau}f_{\mu\kappa}f_{\nu\tau}\int dy\, e^{-\frac{1}{2}nk|y|}(2\pi R)^n(v^0(y))^2. \tag{3.19}$$

This is now finite for constant v^0, and we conclude the theory contains a massless 4D vector boson whose interactions with localised fermions respect gauge invariance.

More complicated scenarios realising similar ideas with > 1 extra dimensions are presented in Refs. [23, 24].

Spin-1/2 fields

Let Ψ be a spin-1/2 fermion field. The Dirac equation in the background of Eq. (2.13)[2] also yields a zero mode $f^0(y)\psi^0(x)$, but this time the extra-dimensional profile is actually exponentially *increasing* away from the brane:

$$f^0(y) = ce^{2k|y|}. \tag{3.20}$$

We can plug this into the action for Ψ to get

$$\begin{aligned}\mathcal{S}_\Psi &= \int dy \int d^4x\sqrt{g}\, i\overline{\Psi}\Gamma^A V_A^N \partial_N \Psi \\ &\supset \int dy\, c^2 e^{k|y|}\int d^4x\, i\overline{\psi}^0(x)\gamma^\alpha\partial_\alpha\psi^0,\end{aligned} \tag{3.21}$$

where the vielbein for the metric (2.13) is given by

$$V_\alpha^N = e^{k|y|}\delta_\alpha^N \quad V_5^N = \delta_5^N. \tag{3.22}$$

[2]See Ref. [3] or appendix B for a review of the Dirac equation in curved spacetime

Clearly the integral over y diverges, so that ψ^0 is not normalisable. Therefore warped gravity cannot localise fermions either[3].

3.4 Gauge boson localisation

We have seen that both fermions and scalars can be localised on a gravitating scalar field domain wall. The final ingredient we need for low-energy model building is the gauge bosons. It has proven to be significantly more difficult to localise gauge bosons in a phenomenologically reasonable way, mainly because it must be insured that gauge invariance is preserved in the low-energy theory. Specifically, the problem that arises is violation of gauge universality, the fact that all charged particles must have the same gauge coupling constant in a non-Abelian gauge theory. To see why this is generically violated by naïve localisation mechanisms, consider the canonical coupling of a gauge field to a fermion:

$$\begin{aligned} \mathcal{S}_c &\sim \int dy \int d^4x \sqrt{g}\, V_C^N \overline{\Psi} \gamma^C \Psi A_N \\ &\supset \int d^4x \left(\int dy\, e^{-3\sigma} |f^0(y)|^2 v^0(y) \right) \overline{\psi}^0 \gamma^\mu \psi^0 a_\mu^0, \end{aligned} \tag{3.23}$$

where V is the vielbein, and f^0 and v^0 are the extra-dimensional profiles of the hypothetical zero modes ψ^0 and a^0 of the fermion and gauge field respectively.

We can see immediately from the above what the problem is: the effective gauge couping in the low-energy theory is determined by an overlap integral of the profiles along the extra dimension. Therefore if different fermions have different localisation profiles, they will generically acquire different gauge couplings in the low-energy theory. The exception to this occurs if v^0 is a constant, in which case the integral that determines the effective coupling constant is proportional to the fermion normalisation integral, and thus equal for all species. We have seen however that a constant mode for a gauge field is not normalisable in the case of a single extra dimension.

[3] In fact, Ref. [21] shows that fermions can be localised on a brane with negative energy density, as this induces an exponentially *increasing* warp factor $e^{2k|y|}$. However, this solution does not localise gravity, so is not physically interesting.

Let us examine some of the suggested mechanisms for gauge boson localisation.

3.4.1 Coupling to a dilaton

In Ref. [17], Kehagias and Tamvakis explore a 5D model with two scalar fields: a field Φ which forms a domain wall, and a second field π, which they call the dilaton. The background action for the theory is

$$\mathcal{S}_{bg} = \int dy \int d^4x\sqrt{g}\left\{-2M^3R + \frac{1}{2}D^M\Phi D_M\Phi + \frac{1}{2}D^M\pi D_M\pi - V(\Phi,\pi)\right\}, \tag{3.24}$$

where M is the 5D gravitational scale, R is the Ricci scalar, and V is some potential which will be chosen to yield the type of background solution required. The ansatz taken for the metric is $ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - e^{-2B(y)}dy^2$. The solution to the Einstein and Klein-Gordon equations is

$$\phi(y) = v\tanh(ky) \tag{3.25}$$

$$\sigma(y) = \beta\log[\cosh^2(ky)] + \frac{\beta}{2}\tanh^2(ky) \tag{3.26}$$

$$\pi(y) = \sqrt{3M^3}\sigma(y), \qquad B(y) = \frac{1}{4}\sigma(y), \tag{3.27}$$

where v and k are constants, and $\beta = v^2/36M^3$.

We now introduce a gauge field A_M, with its field strength tensor F_{MN}, described by the action

$$\mathcal{S}_A = -\frac{1}{4}\int dy\int d^4x\sqrt{g}\,e^{-\lambda\pi/2\sqrt{3M^3}}F_{MN}F^{MN} \tag{3.28}$$

where λ is a dimensionless dilaton coupling constant. The equation of motion for the gauge field again admits a zero mode solution $v^0(y)a_\mu(y)$ where $v^0(y)$ is constant. This time however, it is normalisable. If $f_{\mu\nu}$ again denotes the 4D field strength constructed from a_μ, then we have

$$\begin{aligned}\mathcal{S}_A \supset &-\frac{1}{4}\int d^4x\,\eta^{\mu\kappa}\eta^{\kappa\nu}f_{\mu\nu}f_{\kappa\tau}\int dy\,(v^0)^2e^{-B(y)-\lambda\pi(y)/2\sqrt{3M^3}}\\ = &-\frac{1}{4}\int d^4x\,\eta^{\mu\kappa}\eta^{\kappa\nu}f_{\mu\nu}f_{\kappa\tau}\int dy\,(v^0)^2e^{-(1+2\lambda)A(y)/4}.\end{aligned} \tag{3.29}$$

From the last line we can see that the integral over y is finite, and thus the zero mode is normalisable.

3.4.2 The Dvali-Shifman mechanism

A completely novel way of localising gauge bosons to a domain wall is presented in Ref. [25]. It relies on non-perturbative aspects of quantum non-Abelian gauge theories, specifically the phenomenon of confinement. The gauge symmetry of the theory is $SU(2)$, and the idea is to arrange for $SU(2)$ to be broken to $U(1)$ at the position of a domain wall. The massless $U(1)$ gauge boson should then be confined to the domain wall, which can be argued as follows.

Away from the wall, $SU(2)$ is unbroken, and the theory exhibits confinement. Therefore all states must be at least as massive as Λ_c, the confinement scale of the theory. On the wall, $SU(2)$ is broken to $U(1)$, so two of the gauge bosons will acquire masses. Thus confinement will *not* occur on the wall, and the gauge boson which remains massless will be free to propagate. If this 'photon' is to again propagate off the brane, it must be incorporated into an $SU(2)$ glueball, with mass of order Λ_c. Therefore at low energies, the photon should remain localised on the wall.

This idea is based on work done by Witten in Ref. [26], in the context of cosmic strings.

The original Dvali-Shifman model was in 3+1-dimensional spacetime, but the idea applies in 4+1 dimensions as long as we assume that non-Abelian gauge theories are also confining in 4+1D. As well as the gauge bosons, the theory contains a scalar field χ in the adjoint (triplet) representation of $SU(2)$, and another scalar[4] η which is uncharged under $SU(2)$. It is these scalar fields which will facilitate the symmetry breaking on the brane. The Lagrangian is,

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4g^2}G^a_{\mu\nu}G^{a\mu\nu} + \mathrm{Tr}\left[(D_\mu\chi)^\dagger D^\mu\chi\right] + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta \\ &-\frac{\lambda'}{2}(\mathrm{Tr}\left[\chi^2\right] + \kappa^2 - v^2 + \eta^2)^2 - \lambda(\eta^2 - v^2)^2 \end{aligned} \tag{3.30}$$

where $G^a_{\mu\nu}$ is the field strength tensor, g is the gauge coupling constant, v and κ are mass parameters taken to be much larger than the confinement scale Λ of the $SU(2)$ theory, and λ, λ' are positive dimensionless constants.

[4][25] actually has two fermion doublets as well, but we ignore these here as they don't affect the features we are interested in.

The argument proceeds as follows. The classical minimum of the above potential will have $\eta \simeq \pm v$, and $\chi = 0$. The $SU(2)$ symmetry is unbroken, because η is a singlet under $SU(2)$. But if we ignore initially the coupling to χ, we know that η has a static domain wall solution, interpolating between v and $-v$ at $y = \pm\infty$, where y denotes the extra dimension. We assume that η passes through zero at $y \simeq 0$. We expect a similar solution to persist when the coupling to χ is turned on.

Consider the potential inside the domain wall described above. If $v^2 > \kappa^2$, then when $\eta \simeq 0$, $\chi = 0$ is longer a minimum. Hence we expect χ to develop a non-zero value inside the wall, breaking $SU(2)$ to $U(1)$.

We can be a little more rigorous than this, following the reasoning originally presented in [26]. The equations of motion are satisfied if χ is identically zero and $\eta(y) = v\tanh(my)$ is a kink, where $m = \sqrt{2\lambda}v$. But this can only consistently be used as a background for a quantum theory if it represents a *stable* classical solution [27]. To investigate stability, we consider a perturbation of the third component of χ. Let $\chi_3 = \epsilon\tilde{\chi}e^{-i\omega t}$, where we assume that $\tilde{\chi}$ is a function of y only. Keeping only terms of first order in ϵ, the equation for $\tilde{\chi}$ is:

$$\left[-\frac{\partial^2}{\partial y^2} + \frac{\lambda'}{2}\left(\kappa^2 + v^2(\tanh^2(my) - 1)\right)\right]\tilde{\chi} = \omega^2\tilde{\chi} \tag{3.31}$$

Notice that this is just a time-independent Schrödinger equation. With $\kappa = 0$, the potential is negative definite, and there is known to be a negative-energy bound state solution. By continuity, such a solution exists for some range of non-zero κ. But the 'energy' here is ω^2, so negative energy means ω is imaginary, and thus our perturbation grows exponentially with time. Therefore the solution we started with is unstable, and the stable classical solution must have non-zero χ.

In Ref. [28], results are presented for lattice calculations of a 2+1-dimensional version of the above model. These confirm that a zero mode for the photon is indeed localised to the domain wall, but the resulting dynamics are not truly those of 1+1-dimensional QED. However, the results are expected to change qualitatively and be more promising in the original 3+1-dimensional model. The realistic case of 4+1 dimensions is not discussed, but certainly there are promising signs in this paper that the Dvali-Shifman mechanism works as claimed.

There is an appealing intuitive explanation in Refs. [29, 30] of why this mechanism should work, and also why it should guarantee gauge universality on the brane.

The theory is confining in the bulk, but not so on the wall. By the reasoning of section 4.3, we can think of the wall as being sandwiched between two semi-infinite dual superconductors. The dual Meissner effect will thus exclude field lines of the unbroken gauge group from penetrating the bulk. Instead, the flux will dilute only in the directions parallel to the wall, and the usual 4D Gauss' law will result. Gauge universality is essentially the statement that each particle produces the same amount of flux, so from this picture it is clear that gauge universality in the 5D theory guarantees the same in the 4D theory on the wall. The same applies even if the charge is displaced slightly from the wall; the dual Meissner effect forces the flux to form a string joining the charge to the wall and then spreading out. If a charge is displaced from the wall far enough, a particle-anti-particle pair will be created, and the flux string will break, leaving a single charge confined to the wall, and an $SU(5)$ meson in the bulk.

3.5 Other approaches

Much of the literature on extra dimensions concerns generalisations of the ADD or Randall-Sundrum models, but these are not the only approaches that have been studied. Here we briefly review some of the other major ideas.

3.5.1 Split fermions

In the Standard Model, all fermions gain their mass from Yukawa interactions with the Higgs field. To reproduce the observed masses in this scheme requires a large hierarchy between the coupling constants for the various fields; the top quark coupling constant is approximately 1, but all others are much smaller.

In Ref. [31], Arkani-Hamed and Schmaltz presented a general scheme for naturally generating hierarchical coupling constants in theories with extra dimensions. Their idea is a straightforward generalisation of the localisation of fermions to a domain wall which was presented in section 3.1: Suppose we again have a scalar field η in 4+1 dimensions, and that it assumes a domain wall profile η^{cl} along the extra dimension. Introduce 5D fermion fields Ψ_j, and give each a mass along with its Yukawa coupling to η, so that the fermion action is

$$\mathcal{S}_f = \int d^4x \int dy \sum_j \overline{\Psi}_j [i\Gamma^M \partial_M + h_j \eta - m_j] \Psi_j. \tag{3.32}$$

Recall that in the absence of the mass terms, each chiral zero mode would be localised at the zero of η^{cl}. It is clear that now the chiral zero mode of Ψ_j will instead be localised at the zero of $h_j \eta^{cl} - m_j$. Therefore each fermion species will in fact be localised at a different position in the extra dimension!

We can easily understand how the separation of fermions in the extra dimension generates hierarchical Yukawa couplings at the 4D level. Begin by considering just the first generation leptons. Their 5D action is

$$\begin{aligned} \mathcal{S} = \int d^4x \int dy \big(&\overline{L}[i\Gamma^M \partial_M + h_1 \eta - m_1] L \\ &+ \overline{E}'[i\Gamma^M \partial_M + h_2 \eta - m_2] E' + \kappa \Phi \overline{L^c} E' + \text{h.c.} \big), \end{aligned} \tag{3.33}$$

where L is the lepton doublet, E' the conjugate of the right-handed electron, and Φ the Higgs doublet. Due to their coupling to η, L and E' will have left-handed zero modes $l_L, (e_R)^c$ localised at the zeroes of $h_1 \eta^{cl} - m_1$ and $h_2 \eta^{cl} - m_2$ respectively. The lowest lying mode ϕ of the Φ field will have a profile independent of y (this is only normalisable if the extra dimension is finite). These modes will then give the electron mass term $\lambda_e \phi \overline{(l_L)^c} (e_R)^c + \text{h.c.}$, with the coupling constant λ_e given by the overlap integral of the fermion profiles in the extra dimension. In Ref. [31], it is shown that if the solution for η is linear, $\eta^{cl} = 2\mu^2 y$, where μ is a constant, then the resulting value of λ_e is

$$\lambda_e = \kappa e^{-\mu^2 r^2/2}, \tag{3.34}$$

where r is the separation of the fermions in the extra dimensions. We can see now how hierarchical couplings are naturally generated; the exponential means that even if the separation of left- and right-handed components of different fields differs by only a small amount, their 4D Yukawa couplings will still differ greatly.

Proton decay

This phenomenon also has implications for the problem of proton stability in extensions of the Standard Model. Baryon number is a good approximate low energy symmetry in the Standard Model, and this prevents the proton decaying, because it is the lightest particle carrying non-zero baryon number.

Various extensions of the standard model include baryon number-violating interactions, usually in the form of direct coupling between quarks and leptons. If quarks and leptons are separated in the extra dimension, the coupling constants for these interactions will be exponentially small in the effective 4D theory, thus suppressing the decay rate of the proton to a safe level.

Chapter 4

Miscellaneous

In this chapter we present the main ideas and results of the $SU(5)$ grand unified theory, as well as the field theory techniques which will be most relevant to us later.

4.1 SU(5) grand unification

One of the earliest ideas towards extending the Standard Model was to embed its gauge group $SU(3) \times SU(2) \times U(1)$ in a larger simple Lie group[1]. This general idea goes by the name of grand unification. There are two main advantages that a grand unified theory (GUT) has over the Standard Model: there is only one gauge coupling constant, as opposed to three, and generally fewer representations are needed to contain all the Standard Model particles. The main disadvantage is that extra spontaneous symmetry breaking is required to occur, and this leads to a more complicated Higgs sector.

The earliest and simplest example of a GUT is the $SU(5)$ model of Georgi and Glashow, introduced in Ref. [32]. The Standard Model gauge group is embedded in $SU(5)$ via the mapping $\varphi : SU(3) \times SU(2) \times U(1) \to SU(5)$ given

[1] As distinct from the Standard Model gauge group which is a product of three groups.

by:

$$\varphi(V,U,e^{i\theta}) = \begin{pmatrix} \begin{pmatrix} & & \\ & \mathrm{V} & \\ & & \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \mathrm{U} \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} e^{2\theta} & & & & \\ & e^{2\theta} & & & \\ & & e^{2\theta} & & \\ & & & e^{-3\theta} & \\ & & & & e^{-3\theta} \end{pmatrix} \tag{4.1}$$

The fermions need to be put into representations of $SU(5)$ in such a way that the induced representations of $SU(3) \times SU(2) \times U(1)$ are those of the Standard Model. It turns out that two irreducible representations of $SU(5)$ – the $\mathbf{5}^*$ and the $\mathbf{10}$ – are enough to contain all the Standard Model fermions (and nothing else) in this way. The $\mathbf{5}^*$ is the conjugate of the fundamental representation, and the $\mathbf{10}$ is the anti-symmetric rank 2 tensor, transforming as $\Psi_{10} \to U\Psi_{10}U^T$ for $U \in SU(5)$. The first generation Standard Model particles are assigned to these representations as follows:

$$\mathbf{5}^* : \qquad \begin{pmatrix} (d_{rR})^c & (d_{gR})^c & (d_{bR})^c & e_L & \nu_{eL} \end{pmatrix} \tag{4.2}$$

$$\mathbf{10} : \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & (u_{bR})^c & (-u_{gR})^c & u_{rL} & d_{rL} \\ (-u_{bR})^c & 0 & (u_{rR})^c & u_{gL} & d_{gL} \\ (u_{gR})^c & (-u_{rR})^c & 0 & u_{bL} & d_{bL} \\ -u_{rL} & -u_{gL} & -u_{bL} & 0 & e_L^+ \\ -d_{rL} & -d_{gL} & -d_{bL} & -e_L^+ & 0 \end{pmatrix} \tag{4.3}$$

where subscripts L, R stand for left- and right-handed, r, g, b stand for red, green, and blue, and a superscript c indicates the charge-conjugate wavefunction, $\psi^c = i\gamma^2\psi^*$. Note that charge conjugation also changes the chirality. It is easily checked that these representations break down appropriately under the Standard Model gauge group.

4.1.1 Breaking $SU(5)$

The observed unbroken gauge group of the universe is $SU(3) \times U(1)_Q$, where $U(1)_Q$ is the electromagnetic gauge group. The many successes of the Standard

Model suggest that this arises from the larger group $SU(3) \times SU(2) \times U(1)_Y$ via symmetry breaking at the electroweak scale $M_{ew} \sim 200$GeV. On the other hand, there is no evidence for the existence of the extra vector bosons which occur in an $SU(5)$ theory. Therefore if $SU(5)$ is indeed a symmetry of our universe, it must be broken in two stages: to $SU(3) \times SU(2) \times U(1)_Y$ at a very high energy scale, and then to $SU(3) \times U(1)_Q$ at the electroweak scale.

To facilitate spontaneous symmetry breaking we introduce two scalar fields ϕ_1, ϕ_2 transforming by the **24** (adjoint) and $\mathbf{5}^*$ representations respectively. ϕ_1 induces the first stage of symmetry breaking by developing a very large vev in the direction of the hypercharge generator. The electroweak doublet contained in ϕ_2 then breaks electroweak symmetry as in the Standard Model.

4.1.2 Problems for $SU(5)$ theory

The minimal $SU(5)$ grand unified theory as presented above has some very attractive features, but also brings with it a whole new set of theoretical problems. Here we briefly explain the most important of these problems.

Coupling constant unification

If the Standard Model gauge interactions all arise from the single gauged symmetry $SU(5)$, then they must all share the same coupling constant. The observation at low energies of three very different coupling constants does not necessarily contradict this; as we explain in section 4.2, coupling constants 'run' with energy, and the coupling constants need only be equal above the scale at which $SU(5)$ is broken. We show in section 4.2.2 however that the three coupling constants never become equal in the minimal $SU(5)$ theory.

Proton decay

The extra gauge bosons introduced by the $SU(5)$ symmetry mediate direct interactions between leptons and quarks, and can thus cause the proton to decay. This can however be suppressed sufficiently if the masses of these gauge bosons are large enough i.e. if the scale of $SU(5)$ breaking is large enough. The more serious operators that lead to proton decay involve the colour triplet component ϕ_c of the $\mathbf{5}^*$ scalar, viz. $\overline{u}_R(e_R)^c\phi_c^*$ and $\overline{d}_R(u_R)^c\phi_c$. The existence of these operators creates the...

...doublet-triplet splitting problem

In the $SU(5)$ theory, the electroweak doublet Higgs ϕ_w necessarily comes along with a colour triplet partner ϕ_c which mediates proton decay as described above. Therefore this triplet must be very massive to sufficiently suppress this process. The difficulty arises because the Higgs itself must be light ($\sim$ 200GeV), and thus the two scalars, belonging to the same $SU(5)$ multiplet, must have masses differing by many orders of magnitude. This is difficult to achieve naturally.

The gauge hierarchy problem

This is another problem of vastly differing energy scales. There must be two stages of symmetry breaking in the model, with $SU(5)$ breaking to the Standard Model at a very high scale, and then electroweak symmetry breaking to electromagnetic $U(1)$ at the weak scale. Similarly to the usual hierarchy problem between the electroweak and Planck scale, the mass of the electroweak Higgs should naturally be dragged up to the higher scale by quantum corrections. Avoiding this requires the theory to be fine-tuned.

Fermion masses

In the Standard Model, the Yukawa coupling constants of the various fermion species to the Higgs are entirely independent of each other, and thus can simply be chosen to reproduce the observed masses of these particles. However, in $SU(5)$ the particles are grouped into fewer representations, and thus their Yukawa couplings are related via the $SU(5)$ symmetry. In particular, the down quark and electron each have their right and left-handed parts split between the $\mathbf{5}^*$ and $\mathbf{10}$, which leads at tree level to their Yukawa couplings to the electroweak Higgs being equal. Of course, because this is a tree-level $SU(5)$ relation, it is only valid when $SU(5)$ is unbroken i.e. at and above the unification scale, so to compare to experiment we must run the coupling constants down to low energies. However, there is a relationship between different generations which is unaffected by the running, viz. $m_s/m_d = m_\mu/m_e$ [33]. This grossly contradicts the measured values $m_s/m_d \sim 20$ and $m_\mu/m_e \sim 200$.

4.2 Running coupling constants

In any field theory, each term in the Lagrangian has its own coupling constant. Roughly speaking, these coupling constants tell us how strongly the corresponding fields interact in the classical theory, or at tree level in perturbative quantum field theory. In the quantum theory though, loop diagrams also contribute to scattering amplitudes etc. It is well known that the corrections from these loop diagrams naïvely evaluate to infinity, and must be 'renormalised' to give finite results. The outcome is a finite correction to the coupling constant, which generically depends on the energy scale of the process involved. Therefore in the quantum theory the coupling 'constant' is replaced by a parameter that varies with energy. For historical reasons, this parameter carries the somewhat contradictory name "running coupling constant".

It is conventional to define the "beta function" for a coupling constant λ by

$$\beta_\lambda \equiv \frac{\partial \lambda}{\partial \log M} \tag{4.4}$$

where M is the renormalisation scale at which λ is defined. It is clear that the beta function represents how strongly the running coupling constant depends on the energy scale.

4.2.1 The running coupling in QED

The simplest physically-relevant quantum field theory is quantum electrodynamics (QED), so it is worthwhile presenting explicitly the calculation of the beta function (to one-loop order) in this case. This will also fix our conventions for later one-loop calculations.

Because we will be talking about renormalisation, we need to distinguish between bare fields and parameters and renormalised (physical) ones, so we write the Lagrangian for QED as[2]

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}_0 \gamma^\mu (\partial_\mu - ie_0 A_{0\mu})\psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}, \tag{4.5}$$

[2]We have not included the electron mass. We are only interested in high energy processes, in which the electron mass is negligible.

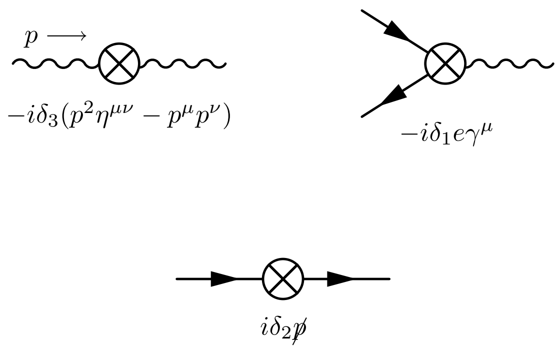

Figure 4.1: Counterterm vertices for QED. Note that the form of the photon self-energy counterterm comes from first expanding $F_{\mu\nu}F^{\mu\nu}$ and integrating by parts.

where a subscript '0' denotes a bare quantity, and other notation is standard. The coupling constant in this theory is the bare charge e_0 of the electron. We will now renormalise this charge at one-loop order. This will require the introduction of a cut-off scale Λ, and a (smaller) renormalisation scale M at which we define the physical quantities. The physical quantities are related to the bare ones via multiplicative constants (we use the notation of Peskin and Schroeder [34]),

$$\psi = Z_2^{-\frac{1}{2}}\psi_0 \quad A_\mu = Z_3^{-\frac{1}{2}}A_{0\mu} \quad e = Z_1^{-1}Z_2 Z_3^{\frac{1}{2}}e_0. \tag{4.6}$$

The renormalisation constants Z_i depend implicitly on the renormalisation scale M and the cut-off Λ. If we now define $\delta_i = Z_i - 1$ for $i = 1, 2, 3$, we can re-write the Lagrangian in terms of the physical quantities along with counter-terms:

$$\begin{aligned}\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi + e\overline{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ + i\delta_2\overline{\psi}\gamma^\mu\partial_\mu\psi + e\delta_1\overline{\psi}\gamma^\mu\psi A_\mu - \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{4.7}$$

The extra Feynman rules which arise from this Lagrangian are shown in Fig. 4.1; the values of the δ_i are chosen to enforce the renormalisation conditions (at each order in perturbation theory).

Before we determine the dependence of e on the scale M at which it is defined, we note a significant simplification, which applies only to Abelian gauge theories. Due to gauge invariance, the relation $Z_1 = Z_2$ always holds, as explained in Ref. [35]. Therefore the expression for e is simply $e = Z_3^{\frac{1}{2}}e_0$. Differ-

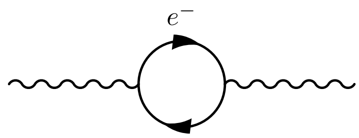

Figure 4.2: The one-loop contribution to vacuum polarisation in QED.

entiating this with respect to[3] $\log M$ gives

$$\frac{\partial e}{\partial \log M} = \frac{e}{2}\frac{\partial \log Z_3}{\partial \log M}. \tag{4.8}$$

Finally, we expand $\log Z_3$:

$$\log Z_3 = \log(1+\delta_3) = \delta_3 + \mathcal{O}(\delta_3^2). \tag{4.9}$$

We will see that at one-loop order δ_3 is proportional to e^2, so to lowest order we need only keep the first term of this expansion. Thus we will take our one-loop beta function to be

$$\beta_e = \frac{e}{2}\frac{\partial \delta_3}{\partial \log M}. \tag{4.10}$$

Our task then is to calculate the counter-term δ_3, corresponding to renormalisation of the photon propagator, at one-loop order. The only Feynman diagram to consider has a single electron loop, as shown in Fig. 4.2. Applying the Feynman rules for QED, the amplitude for this diagram is[4]

$$(ie)^2 \int \frac{d^4 l}{(2\pi)^4} (-1) \mathrm{Tr}\left[\frac{i\not{l}}{l^2}\gamma^\mu \frac{i(\not{l}-\not{p})}{(l-p)^2}\gamma^\nu\right] \tag{4.11}$$

The standard techniques (Feynman parameters etc.) for evaluating such integrals are well known, and described in e.g. Ref. [34]. We use dimensional regularisation, working in $d = 4 - \epsilon$ dimensions. The above integral eventually

[3]It will become clear later why we differentiate with respect to $\log M$ rather than M.

[4]We in fact consider the ‘amputated’ diagram, meaning we don’t include the external propagators.

becomes[5]

$$-i\frac{e^2}{2\pi^2}(p^2\eta^{\mu\nu}-p^\mu p^\nu)\int dx\, x(1-x)\left\{\frac{2}{\epsilon}-\log[-x(1-x)p^2]-\log\pi-\gamma+\mathcal{O}(\epsilon)\right\}, \tag{4.12}$$

where γ is the Euler-Mascheroni constant. We renormalise at large spacelike momentum, and thus let $p^2 = -M^2$. We can then perform the integral over x, and get

$$i\frac{e^2}{12\pi^2}(p^2\eta^{\mu\nu}-p^\mu p^\nu)\log M^2 + \text{M-independent terms.} \tag{4.13}$$

We ignore the M-independent terms because they do not contribute to the (one-loop) beta function. The δ_3 counterterm must cancel with this amplitude, so we require

$$\delta_3 = \frac{e^2}{12\pi^2}\log M^2 + \text{M-independent terms.} \tag{4.14}$$

Finally then, we obtain the beta function:

$$\beta_e = \frac{e^3}{12\pi^2}. \tag{4.15}$$

We notice one thing immediately: this is positive, implying that the strength of the electromagnetic interaction increases at large energies. An explicit solution for the coupling constant is usually given in terms of the fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$. First recast the differential equation in terms of α:

$$\frac{de}{d\log M} = \beta_e = \frac{e^3}{12\pi^2} \quad\Longrightarrow\quad \frac{d\alpha}{d\log M} = \frac{2\alpha^2}{3\pi}. \tag{4.16}$$

There will be a single integration constant to fix. We do this by measuring α at a reference energy $\overline{M}$, and defining $\overline{\alpha} = \alpha(\overline{M})$. The solution to the above is then

$$\alpha(M) = \frac{\overline{\alpha}}{1-\frac{\overline{\alpha}}{3\pi}\log\left(\frac{M^2}{\overline{M}^2}\right)}. \tag{4.17}$$

4.2.2 Coupling constant unification

The Standard Model is a gauge theory with three different gauge coupling constants. The three corresponding beta functions receive contributions from different particles, and thus the three constants run differently with energy. As it

[5] The appearance of the logarithm of a dimensionful quantity is unusual. This is an artifact of the dimensional regularisation technique; such nonsensical expressions never appear in physical quantities.

turns out, they become approximately equal at an energy of 10^{13-17}GeV. We will now sketch the derivation of this very important result, which is crucial to the ongoing interest in grand unified theories.

The one-loop beta functions of the three gauge couplings are independent of the others, so we can calculate the running separately. For a non-Abelian gauge theory, it is not true that $Z_1 = Z_2$, so the analogue of Eq. (4.8) is

$$\begin{aligned}\frac{\partial g}{\partial \log M} &= g\frac{\partial}{\partial \log M}\left(-\log Z_1 + \log Z_2 + \frac{1}{2}\log Z_3\right)\\ &\simeq g\frac{\partial}{\partial \log M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right),\end{aligned} \tag{4.18}$$

where the second equality again holds only to first order. Our task then is to calculate the counterterms to one-loop order. For reference, we record the first generation standard model fields with their representations under $SU(3) \times SU(2) \times U(1)_Y$:

$$\begin{aligned}l_L \sim (1,2,-1) \quad Q_L \sim (3,2,\frac{1}{3})\\ e_R \sim (1,1,-2) \quad u_R \sim (3,1,\frac{4}{3}) \quad d_R \sim (3,1,-\frac{2}{3})\end{aligned} \tag{4.19}$$

We also have the Higgs boson ϕ which transforms as $\phi \sim (1,2,1)$. In appendix C, we demonstrate the well-known result that for an $SU(N)$ gauge theory, the one-loop beta function is given by

$$\beta(g) = -\frac{g^3}{48\pi^2}[11N - \sum_f 2C(\mathbf{r}_f) - \sum_s C(\mathbf{r}_s)], \tag{4.20}$$

where f labels the *Weyl* fermions in the theory, $\mathbf{r}_f$ their representations, and s labels the *complex* scalars, with $\mathbf{r}_s$ their representations. The quantity $C(\mathbf{r})$ associated with a representation $\mathbf{r}$ is defined in section 1.0.1. The above expression is valid when all masses are much smaller than the energy scale we are considering, but as a good approximation we can treat all particles that are lighter than the relevant scale as massless, and all those heavier as infinitely massive. The standard model coupling constants are accurately measured at the mass of the Z boson, M_Z, so we can begin the running from there. Therefore the massless approximation works well for all particles except the Higgs and the top quark. However these particles have masses of the same order as M_Z, so for simplicity we will simply assume that everything is massless. This should give a reasonable first approximation, especially as the coupling constants only run logarithmically, as we will see, and thus change little between M_Z and M_t.

The one-loop beta functions are all proportional to the coupling constant cubed, so that the equation we have to solve is

$$\frac{dg}{d\log M} = \frac{b}{8\pi^2}g^3. \tag{4.21}$$

Usually the solution is instead given for the fine structure constant for the theory, defined by $\alpha = g^2/4\pi$, in terms of which the above becomes

$$\frac{d\alpha}{d\log M} = \frac{b}{\pi}\alpha^2. \tag{4.22}$$

Incorporating the initial condition of the measured value at M_Z, the solution to this equation is

$$\alpha^{-1}(M) = \alpha^{-1}(M_Z) - \frac{b}{2\pi}\log\left(\frac{M}{M_Z}\right)^2 \tag{4.23}$$

Now all we need to do is calculate the co-efficient b for each factor of the Standard Model gauge group.

$\underline{SU(3)}$

For $SU(3)$, the gauge group of the strong interaction, we have 4 Weyl quarks per generation (the electroweak doublet, plus the right handed pair of each component of the doublet), all in the fundamental **3** representation. We have by definition $C(\mathbf{3}) = \frac{1}{2}$, so the beta function for the $SU(3)$ coupling constant is

$$\begin{aligned}\beta(g_3) &= -\frac{g_3^3}{48\pi^2}\left[33 - 12\right] \\ &= -\frac{7}{16\pi^2}g_3^3\end{aligned} \tag{4.24}$$

Therefore we have for the strong interaction

$$\alpha_3^{-1}(M) = \alpha_3^{-1}(M_Z) + \frac{7}{4\pi}\log\left(\frac{M}{M_Z}\right)^2. \tag{4.25}$$

$\underline{SU(2)}$

For $SU(2)$, the leptons contribute a single fundamental **2** representation per generation, as does each colour of left-handed quark doublet. There is furthemore the Higgs scalar in the fundamental representation, and again we have $C(\mathbf{2}) = \frac{1}{2}$, so that

$$\begin{aligned}\beta(g_2) &= -\frac{g^3}{48\pi^2}\left[22 - 12\times\left(2\times\frac{1}{2}\right) - \frac{1}{2}\right] \\ &= -\frac{19}{96\pi^2}g_2^3,\end{aligned} \tag{4.26}$$

which leads to

$$\alpha_2^{-1}(M) = \alpha_2^{-1}(M_Z) + \frac{19}{24\pi}\log\left(\frac{M}{\overline{\overline{M}}}\right)^2 . \tag{4.27}$$

<u>$U(1)_Y$</u>

The coupling constant of a gauge theory depends on our choice of normalisation of the generator(s). For non-Abelian groups we conventionally choose $\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$, but for an Abelian group there is no easy way to define such a convention. Therefore we need to decide how to normalise the hypercharge generator to determine whether or not the coupling constants unify. Given that we discussed $SU(5)$ grand unification in the last section, the way to proceed is obvious: we will embed hypercharge in $SU(5)$ and determine the appropriate normalisation that way. The $SU(5)$ generators are conventionally normalised so that $\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$. From the fact that the right-handed down quark and the lepton doublet are in the $\mathbf{5}^*$ representation of $SU(5)$ and have hypercharge as given by Eq. (4.19), we see that Y is given by

$$Y = \begin{pmatrix} \frac{2}{3} & & & & \\ & \frac{2}{3} & & & \\ & & \frac{2}{3} & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}, \tag{4.28}$$

and thus that the appropriately normalised generator of $SU(5)$ is in fact $\sqrt{3}Y/2\sqrt{5}$. Therefore the coupling constant g_1 which we might expect to unify with the other coupling constants is given in terms of the hypercharge coupling constant

g_Y by

$$\begin{aligned} g_1 \frac{1}{2}\sqrt{\frac{3}{5}} Y &= g_Y Y \\ \Rightarrow g_1 &= 2\sqrt{\frac{5}{3}} g_Y . \end{aligned} \tag{4.29}$$

Therefore the beta function we want to calculate is $\beta(g_1)$, which is

$$\beta(g_1) = \frac{dg_1}{d\log M} = 2\sqrt{\frac{5}{3}}\frac{dg_Y}{d\log M} = 2\sqrt{\frac{5}{3}} b_Y g_Y^3 = \frac{3}{20} b_Y g_1^3. \tag{4.30}$$

Therefore we need to calculate the constant b_Y for the function $\beta(g_Y)$. Instead of the quantity $C(\mathbf{r})$, each particle simply contributes its hypercharge squared (which can easily be seen by revisiting the calculation of the QED beta function). Each fermion generation contains 15 Weyl fermions in total, with 2 in l_L, 6 in Q_L, 1 in e_R, 3 in u_R and 3 in d_R, with their corresponding hypercharge. Then there is the Higgs, containing two complex scalar fields, so that we get

$$\begin{aligned} \beta(g_Y) = \frac{g_Y^3}{48\pi^2}\Bigg[6 \times \Bigg\{2 \times (-1)^2 + 6 \times \left(\frac{1}{3}\right)^2 + (-2)^2 + 3 \times \left(\frac{4}{3}\right)^2 \\ + 3\left(-\frac{2}{3}\right)^2\Bigg\} + 2 \times (1)^2\Bigg] \\ = \frac{41}{24\pi^2} g_Y^3, \end{aligned} \tag{4.31}$$

and thus

$$\beta(g_1) = \frac{41}{160\pi^2} g_1^3. \tag{4.32}$$

Finally then we have the running fine structure constant for the hypercharge interaction:

$$\alpha_1^{-1}(M) = \alpha_1^{-1}(M_Z) - \frac{41}{40\pi}\log\left(\frac{M}{M_Z}\right)^2 \tag{4.33}$$

We now simply need to insert the experimentally measured values of each coupling constant at M_Z. The quantities which are actually measured are the electromagnetic fine structure constant α_{em}, the weak mixing angle $\sin\theta_W$ and the strong fine structure constant α_3, which are given in Ref. [33] as (we will ignore experimental errors)

$$\alpha_{em}^{-1}(M_Z) = 127.906, \quad \sin^2\theta_W(M_Z) = .2312, \quad \alpha_3(M_Z) = .1187. \tag{4.34}$$

The relationships we need to use to extract α_1, α_2 from the above values are

$$e = g_2 \sin\theta_W, \quad sin^2\theta_W = \frac{(g')^2}{g_2^2 + (g')^2}, \tag{4.35}$$

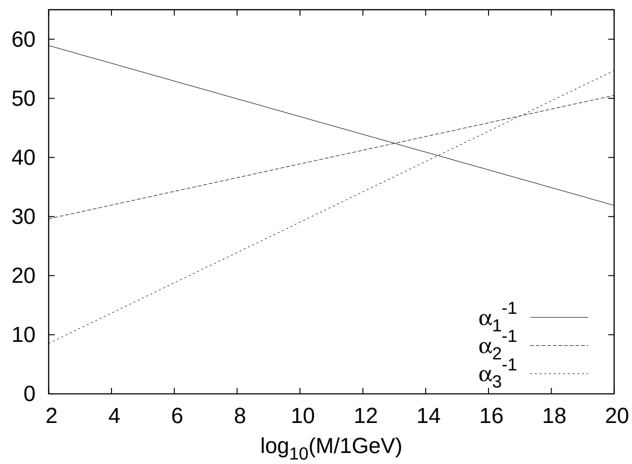

Figure 4.3: The running of the Standard Model coupling constants, calculated to one-loop order, showing approximate unification at $\sim 10^{15}$GeV.

where $g' = 2g_Y$. From this we easily obtain

$$\alpha_1^{-1}(M_Z) = 59.00, \quad \alpha_2^{-1}(M_Z) = 29.57, \quad \alpha_3^{-1}(M_Z) = 8.425, \tag{4.36}$$

and thus obtain the graph shown in figure 4.3. It can be seen that the coupling constants are never all equal, but do approximately meet at an energy of around 10^{15}GeV. Because we performed a very rough calculation, we may hope that taking into account the actual masses of the standard model particles as well as higher-order contributions to the beta functions would result in true unification. Unfortunately this is not true, and the minimal $SU(5)$ model is ruled out by this failure of coupling constant unification[6].

4.3 Confinement in non-Abelian gauge theories

Although the fundamental constituents of nuclear matter are believed to be quarks and gluons, these particles are never observed singly; they always form particles which are uncharged under the $SU(3)$ gauge group of quantum chromodynamics (QCD). This leads to the concept of "confinement", which states

[6]Our calculation did not include the extra scalars that comes from the **24** and $\mathbf{5}^*$ representations, nor the extra $SU(5)$ gauge bosons. This is because these would all gain masses at the scale of $SU(5)$ breaking, which must be of the same order as the unification scale.

roughly that all physical states must be singlets under $SU(3)$ (but see Ref. [36] for possible subtleties in the definition).

Although confinement has not been proven to occur in an $SU(3)$ gauge theory, there is strong numerical evidence that this is the case. When the static potential between a quark and its anti-quark is calculated, a piece is found which rises linearly with distance. This results in a constant attractive force at large distances, and thus an infinite 'binding energy' (see eg. Ref. [37]). This behaviour occurs because the gauge field flux concentrates in a thin 'string', of constant energy density, joining the two quarks. Clearly the energy contained is such a string is proportional to its length.

There is a complementary phenomenon to confinement which *can* be derived rigorously from QCD (in fact we have done so in the last section), called asymptotic freedom. This was discovered independently in the 70s by Politzer, and by Gross and Wilczek, and refers to the fact that the running coupling constant in a non-Abelian gauge theory approaches zero at large energies [38, 39]. The flip-side of this is that the coupling constant is larger at lower energies, corresponding to larger distances. For QCD, the coupling constant becomes of order 1 at an energy of $\sim$ 200MeV, so below this perturbation theory is useless. It is thought that non-perturbative effects in this regime are responsible for confinement. Because all non-Abelian gauge theories exhibit asymptotic freedom[7], and confinement has been demonstrated numerically on the lattice for an $SU(3)$ theory (also for $SU(2)$; see Ref. [40]), it is assumed that all non-Abelian gauge theories exhibit confinement.

Dual superconductivity and confinement

There is one way of understanding confinement without resorting to numerics; it is known as the dual superconductor picture, and was suggested independently by 't Hooft and Mandelstam [41, 42]. Before we discuss this idea, we need to quickly review the conventional understanding of ordinary superconductivity.

In a superconductor, electrons form spin-0 pairs called Cooper pairs. These

[7]In fact, asymptotic freedom can be destroyed if enough different types of fermions are included in the theory. This is not usually relevant though.

spin-0 pairs constitute an effective scalar field, which at low energy (temperature) gains a vacuum expectation value and thus breaks electromagnetic gauge invariance.

One of the characteristic features of superconductivity is the Meissner effect, which refers to the exclusion of magnetic flux from the superconductor. Because of this phenomenon, if two oppositely charged magnetic monopoles are placed in the bulk of the superconductor, the magnetic field lines form a thin tube joining the two monopoles. It is precisely this aspect of superconductivity which motivates the dual superconductor picture of confinement.

A generic feature of non-Abelian gauge theories is the occurrence of magnetic monopoles, which carry a conserved magnetic charge corresponding to a magnetic $U(1)$ symmetry. It is conceivable that in the vacuum these magnetic monopoles could condense and break the magnetic $U(1)$, just like condensed Cooper pairs break the electric $U(1)$ in a superconductor. The formation of such a magnetic condensate should thus lead to a dual Meissner effect, in which electric flux is confined to thin strings joining charges. This is precisely what we need to explain confinement. For a thorough review of this idea, see Ref. [43].

4.4 Charge conjugation

If we are going to consider field theories in dimensions other than four, we need to know how to implement operations such as charge conjugation. Suppose we have a fermion field Ψ in 5D. The Dirac algebra can still be realised by 4×4 matrices, so that Dirac spinors have four components, and the gamma matrices are given by

$$\Gamma^\mu = \gamma^\mu, \qquad \Gamma^5 = -i\gamma^5, \tag{4.37}$$

where γ^μ and γ^5 are the usual 4D Dirac matrices and chirality operator respectively. The Dirac equation for a fermion of charge q coupled to a gauge field A_M is given by

$$i\Gamma^M(\partial_M - iqA_M)\Psi = 0, \tag{4.38}$$

whereas we want the charge conjugate spinor field Ψ^c to satisfy the same equation with opposite charge:

$$i\Gamma^M(\partial_M + iqA_M)\Psi^c = 0. \tag{4.39}$$

To determine the charge conjugation operator, we need to manipulate Eq. (4.38) until it matches Eq. (4.39). To proceed we need to specify a particular representation of the gamma matrices. We choose to use the Dirac-Pauli representation, so that we have

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{4.40}$$

where σ^i are the Pauli spin matrices and $\mathbf{1}$ is the 2×2 identiy matrix. We can see that in this representation, Γ^2 and Γ^5 are purely imaginary, while the other gamma matrices are real.

In the Dirac-Pauli representation then, if we take the complex conjugate of Eq. (4.38) and multiply it on the left by $-\Gamma^5\Gamma^2$, we get

$$i\Gamma^M(\partial_M + iqA_M)\Gamma^5\Gamma^2\Psi^* = 0. \tag{4.41}$$

Comparing with Eq. (4.39), we conclude that (up to an arbitrary phase), $\Psi^c = \Gamma^5\Gamma^2\Psi^*$.

Via a much simpler but analogous calculation, it can be checked that charge conjugation for a scalar field is realised simply by complex conjugation, regardless of the dimensionality of spacetime.

Part II

Non-compact extra dimensions: generalities

Chapter 5

Smoothed Randall-Sundrum-like gravity

We eventually want to construct a purely field theoretical model in five non-compact dimensions. In this context, the only mechanism by which 4D gravity is known to be reproduced is that of the type 2 Randall-Sundrum model [10]. This model contains a delta-function brane in the fundamental action, which is contrary to our philosophy of dynamically localising fields, so we must first ask to what extent the graviton localisation of Ref. [10] can be reproduced by a field theory of the type mentioned in section 3.2.1.

5.1 Scalar field/gravity systems

Consider a 5D model containing a number of scalar fields Φ_j with potential $V(\Phi)$. These will form our classical background. To include gravity, we simply add the Einstein-Hilbert term to the action, and minimally couple other fields to gravity as usual, to obtain the background action

$$\mathcal{S}_{bg} = \int d^4x \int dy \, \sqrt{g} \Big\{ -2M^3 R - \Lambda + \frac{1}{2} g^{MN} \partial_M \Phi_j \partial_N \Phi_j - V(\Phi) \Big\}, \qquad (5.1)$$

where g_{MN} is the 5D metric, g its determinant, M the 5D gravitational mass scale, R the 5D Ricci scalar, and Λ the bulk cosmological constant. The only additional assumptions we make are that V has a $\mathbb{Z}_2$ symmetry $\Phi_j \to -\Phi_j \; \forall \, j$, so that the global minimum of V is at least doubly degenerate, and attained for (say) $\Phi_j = \pm\Phi_j^{\rm min}$. We also require this $\mathbb{Z}_2$ to be independent of any continuous symmetries of the theory.

Now we must solve the coupled Einstein and Klein-Gordon equations associated with the above action. We assume that the metric is a smooth version of that in the RS2 model, $ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$, and the scalar solution $\phi_j(y)$ depends only on y, and satisfies the boundary conditions $\phi_j \to \pm\Phi_j^{\rm min}$ as $y \to \pm\infty$[1]. The resulting Einstein equations are

$$\sigma'' = \frac{1}{12M^3}\sum_j \left(\frac{d\Phi_j}{dy}\right)^2 \tag{5.2}$$

$$(\sigma')^2 = \frac{1}{24M^3}\left\{-\Lambda + \frac{1}{2}\sum_j \left(\frac{d\Phi_j}{dy}\right)^2 - V(\eta,\chi)\right\}, \tag{5.3}$$

while the Klein-Gordon equations are,

$$\frac{d^2\Phi_j}{dy^2} - 4\sigma'\frac{d\Phi_j}{dy} = \frac{\partial V}{\partial \Phi_j}. \tag{5.4}$$

This set of equations has been solved by various authors [14, 15, 44, 19, 17, 18] and here we need only that the resulting warp factor σ is a smooth even function of y, and $\sigma \to \mu|y|$ as $|y| \to \infty$, where μ is some mass scale. In the RS2 model, the domain-wall is instead a delta-function brane, and $\sigma(y) = \mu|y|$ everywhere.

If 4-dimensional Newtonian gravity is reproduced by this model, it will be mediated by fluctuations of the purely 4-dimensional part of the metric. As such, we will consider a metric perturbation of the form

$$ds^2 = \big(e^{-2\sigma(y)}\eta_{\mu\nu} + h_{\mu\nu}(x,y)\big)dx^\mu dx^\nu - dy^2. \tag{5.5}$$

If we substitute this into Einstein's equations and expand to first order in h, the h-independent terms all cancel because our background solution satisfies the

[1]Such a solution is topologically stable, and can be used as a classical background for a quantum field theory [27].

equations. If we choose transverse-traceless gauge, in which $\partial^\mu h_{\mu\nu} = h^\mu_\mu = 0$, we are left with

$$\frac{1}{2}\partial_y^2 h_{\mu\nu} - \frac{1}{2}e^{2\sigma}h_{\mu\nu,\kappa}{}^{\kappa} - h_{\mu\nu}\left[8(\sigma')^2 - 4\sigma'' + \frac{1}{4M^3}\left(\frac{1}{2}\sum_i(\Phi_i')^2 + \Lambda + V(\Phi_i)\right)\right] = 0. \tag{5.6}$$

We can again apply the Einstein equations satisfied by our background solution to reduce this to

$$\frac{1}{2}\partial_y^2 h_{\mu\nu} - \frac{1}{2}e^{2\sigma}h_{\mu\nu,\kappa}{}^{\kappa} - h_{\mu\nu}[2(\sigma')^2 - \sigma''] = 0. \tag{5.7}$$

This is the equation that determines the 4D graviton spectrum. We have already obtained a nice result: this equation does not explicitly depend on the background configuration of the scalar fields. As long as the combined Einstein and Klein-Gordon equations are satisfied, the graviton spectrum can be analysed knowing only the warped metric.

We now need to solve Eq. (5.7). By linearity, we can consider just a single graviton mode of definite 4D mass m, by defining $h_{\mu\nu}(x,y) = f(y)\hat{h}_{\mu\nu}(x)$ where $\hat{h}_{\mu\nu,\kappa}{}^{\kappa} = -m^2\hat{h}_{\mu\nu}$. This yields the following equation for f:

$$-\frac{1}{2}f'' + [2(\sigma')^2 - \sigma'']f - \frac{m^2e^{2\sigma}}{2}f = 0. \tag{5.8}$$

We can follow the lead of Ref. [10] by defining $f = e^{-\frac{1}{2}\sigma}\tilde{f}$ and changing coordinates to z such that $dz/dy = e^\sigma$. The resulting equation for $\tilde{f}$ is then just a Schrödinger equation:

$$-\frac{1}{2}\frac{d^2\tilde{f}}{dz^2} + \left[\frac{9}{8}\left(\frac{d\sigma}{dz}\right)^2 - \frac{3}{4}\frac{d^2\sigma}{dz^2}\right]\tilde{f} \quad = \quad \frac{m^2}{2}\tilde{f} \tag{5.9}$$

$$\Rightarrow \left[-\frac{d}{dz} + \frac{3}{2}\frac{d\sigma}{dz}\right]\left[\frac{d}{dz} + \frac{3}{2}\frac{d\sigma}{dz}\right]\tilde{f} \quad = \quad m^2\tilde{f}. \tag{5.10}$$

The second line demonstrates that the equation is of the form $A^\dagger A\tilde{f} = m^2\tilde{f}$, where the adjoint is defined with respect to the usual inner product on $L^2(\mathbb{R})$[2]. This guarantees that all eigenvalues m^2 are non-negative[3]. A negative eigenvalue

[2]This is the space of square integrable functions on the real line ie. physical solutions of Eq. (5.9). The inner product referred to is the one familiar from 1D quantum mechanics.

[3]To prove this, simply take the inner product with f to get $< f, A^\dagger Af >= m^2 < f, f >$, and then note from the definition of the adjoint that $< f, A^\dagger Af >=< Af, Af >$.

would correspond to a tachyonic graviton mode, signalling an instability in the solution, so what we have proven is that all such models are free of instabilities in the 4D graviton sector.

Equally important as stability is the existence of a solution of Eq. (5.9) with $m = 0$, which can then be identified with the usual 4D graviton. The easiest way to find this solution is to note that if we wish to solve Eq. (5.10) with $m = 0$, we can instead solve the much simpler equation

$$\left[\frac{d}{dz} + \frac{3}{2}\frac{d\sigma}{dz}\right]\tilde{f} = 0, \tag{5.11}$$

which has the solution[4]

$$\tilde{f}(y) = Ce^{-\frac{3}{2}\sigma(y)} \implies f(y) = Ce^{-2\sigma(y)}. \tag{5.12}$$

Therefore the theory indeed contains a normalisable massless mode $h^0_{\mu\nu}(x, y) = Ce^{-2\sigma(y)}\hat{h}^0_{\mu\nu}(x)$, which can mediate normal 4D (perturbative) gravity. Notice that the profile of this mode in the extra dimension is precisely the warp factor from our solution for the metric. This will be important below when we verify that the 4D equivalence principle is respected by these theories.

Most of the results presented in the preceding section were obtained in Ref. [1], but were rediscovered independently by the author, and thus have been presented here in a different fashion.

5.2 Identifying the 4D Planck mass

We can easily identify the familiar Planck mass in terms of M and our solution. To do this we let the 4D part of the Randall-Sundrum metric vary from the Minkowski metric, so we have $ds^2 = e^{-2\sigma(y)}\overline{g}_{\mu\nu}(x)dx^\mu dx^\nu - dy^2$. If we let $\overline{R}$ denote the 4D Ricci scalar corresponding to $\overline{g}$, then the 5D Einstein-Hilbert

[4]Of course Eq. (5.9) is second-order, so we expect two linearly independent solutions. The second one is not normalisable, and therefore not physical.

action yields a 4D Einstein-Hilbert term as follows:

$$\begin{aligned}
\mathcal{S} &= -\int d^4x \int dy \sqrt{g}\, 2M^3 R \\
&\supset -\int d^4x \int dy \sqrt{g}\, 2M^3 e^{2\sigma(y)} \overline{R} \\
&= -\int d^4x \sqrt{\overline{g}} \Big(\int dy\, 2M^3 e^{-2\sigma(y)} \Big) \overline{R} \\
\Rightarrow M_{Pl}^2 &= \int dy\, 2M^3 e^{-2\sigma(y)}
\end{aligned} \tag{5.13}$$

This is exactly analogous to the Randall-Sundrum model, but it is worth at this stage actually describing explicitly how the 5D Ricci scalar 'contains' the 4D one multiplied by a factor of $e^{2\sigma}$. This is really quite simple: the Riemann tensor can be written schematically as

$$\begin{aligned}
R^M{}_{NPQ} &\sim \partial\Gamma + (\Gamma)^2 \\
&\sim \partial(g^{-1}\partial g) + (g^{-1}\partial g)^2,
\end{aligned} \tag{5.14}$$

where Γ represents the Christoffel symbols, and the Ricci tensor is just the contraction of this over the indices M, P. To obtain the '4D part' of the Ricci tensor we can simply throw away any terms in which any index takes the value '5'. In the remaining terms, powers of $e^{2\sigma}$ appear simply as multiplicative constants, and it can be seen that they will cancel between factors of g and g^{-1}. The Ricci scalar then comes from contracting the Ricci tensor with the inverse metric; this introduces the overall factor of $e^{2\sigma}$.

5.3 The equivalence principle

The existence of a massless graviton is not quite enough to guarantee that we reproduce 4D gravity; we must also demonstrate that it couples with equal strength to all stress-energy sources, so as to respect the weak equivalence principle.

In the full 5D theory, the graviton couples to matter through the simple Lagrangian $\sqrt{G_N^{(5)}} h_{MN} T^{MN} = \sqrt{G_N^{(5)}} g^{MK} g^{NL} h_{MN} T_{KL}$, where $G_N^{(5)}$ is the 5D Newton's constant. We need to show that integrating this term over the extra dimension yields the same result for any reasonable stress-energy tensor.

The stress-energy tensor of a theory can be found by varying the matter part of the action with respect to the (inverse) metric. In an arbitrary n-dimensional spacetime, we get

$$\begin{aligned}\delta\mathcal{S}_m &= \delta\left(\int d^n x\sqrt{g}\mathcal{L}_m\right) \\ &= \int d^n x\big[\sqrt{g}\delta\mathcal{L}_m + (\delta\sqrt{g})\mathcal{L}_m\big] \\ &= \int d^n x\sqrt{g}\left(\frac{\delta\mathcal{L}_m}{\delta g^{MN}} - \frac{1}{2}g_{MN}\mathcal{L}_m\right)\delta g^{MN}, \end{aligned} \tag{5.15}$$

and thus identify the stress-energy tensor (up to a constant multiplicative factor):

$$T_{MN} \propto \frac{\delta\mathcal{L}_m}{\delta g^{MN}} - \frac{1}{2}g_{MN}\mathcal{L}_m. \tag{5.16}$$

Therefore if g_{MN} represents our background solution, the action governing the coupling between graviton and matter in our 5D theory is

$$\mathcal{S}_G \propto \int d^4x\int dy\sqrt{g^{(5)}}g^{MN}g^{KL}h_{MK}\left(\frac{\delta\mathcal{L}_m}{\delta g^{NL}} - \frac{1}{2}g_{NL}\mathcal{L}_m\right). \tag{5.17}$$

We note that for our solution, $g^{(5)} = e^{-4\sigma}$, and so does not depend on x. For current purposes we are interested only in the 4D part of this interaction ie. when $M, N = \mu, \nu \in \{0, 1, 2, 3\}$ and we have integrated out the extra dimension. The relevant extra-dimensional profiles are given by equations (7.10) and (5.12):

$$g_{\mu\nu}(x,y) = e^{-2\sigma(y)}\eta_{\mu\nu} \tag{5.18}$$

$$g^{\mu\nu}(x,y) = e^{2\sigma(y)}\eta^{\mu\nu} \tag{5.19}$$

$$h^0_{\mu\nu}(x,y) = Ce^{-2\sigma(y)}\hat{h}^0_{\mu\nu}(x). \tag{5.20}$$

We will treat the two terms in (5.17) separately. The first term can be dealt with in a slightly sneaky way. For our solution, we have $g^{\mu\nu}(x,y) = e^{2\sigma(y)}\eta^{\mu\nu}$; therefore instead of differentiating with respect to $g^{\mu\nu}$, we can get the same result by formally differentiating with respect to $\eta^{\mu\nu}$ and multiplying by $e^{-2\sigma}$. Using this trick, the first term of (5.17) is

$$\int d^4x\int dy\sqrt{g^{(5)}}g^{\mu\nu}g^{\kappa\tau}h^0_{\mu\kappa}e^{-2\sigma}\frac{\delta\mathcal{L}_m}{\delta\eta^{\nu\tau}} \tag{5.21}$$

$$= C\int d^4x\, e^{2\sigma}\eta^{\mu\nu}e^{2\sigma}\eta^{\kappa\tau}e^{-2\sigma}\hat{h}^0_{\mu\kappa}e^{-2\sigma}\frac{\delta}{\delta\eta^{\nu\tau}}\int dy\sqrt{g^{(5)}}\mathcal{L}_m \tag{5.22}$$

$$= C\int d^4x\,\eta^{\mu\nu}\eta^{\kappa\tau}\hat{h}^0_{\mu\kappa}\frac{\delta}{\delta\eta^{\nu\tau}}\mathcal{L}^{(4)}_m. \tag{5.23}$$

where the dimensionally reduced matter Lagrangian $\mathcal{L}_m^{(4)}$ is defined, regardless of the details of the theory, as

$$\mathcal{L}_m^{(4)} = \int dy \sqrt{g^{(5)}} \mathcal{L}_m. \tag{5.24}$$

The second term is very straightforward, and gives, after cancelling powers of e^σ as above,

$$-\frac{C}{2} \int d^4x \, \eta^{\mu\nu} \eta^{\kappa\tau} \hat{h}^0_{\mu\kappa} \eta_{\nu\tau} \mathcal{L}_m^{(4)}. \tag{5.25}$$

Therefore the full dimensionally reduced action corresponding to Eq. (5.17) is given by

$$\mathcal{S}_G \propto \int d^4x \, \eta^{\mu\nu} \eta^{\kappa\tau} \hat{h}^0_{\mu\kappa} \left(\frac{\delta \mathcal{L}_m^{(4)}}{\delta \eta^{\nu\tau}} - \frac{1}{2} \eta_{\nu\tau} \mathcal{L}_m^{(4)} \right). \tag{5.26}$$

We can see by comparison with Eq. (5.17) that this is exactly the action we would have written down for an ordinary 4D graviton in Minkowski space-time coupled to matter described by $\mathcal{L}_m^{(4)}$. Thanks to the fact that the extra-dimensional profile of $h^0_{\mu\nu}$ is proportional to the warp factor, our argument was independent of any details of the fields contributing to $\mathcal{L}_m^{(4)}$, including their extra-dimensional profiles. We can conclude that the 4D equivalence principle is obeyed by all models of the type under consideration.

5.4 Corrections to Newton's law

There is yet another aspect of gravity in the smooth Randall-Sundrum scenario that we need to comment on. From Eq. (5.9), and the fact that asymptotically we have $\sigma \sim |y| \sim \log |z|$, we can see that the effective potential seen by the gravitons asymptotes to zero. From elementary quantum mechanics, we know that this means it admits a continuum of solutions with all positive energies. In other words, our effective 4D theory will contain a continuum of massive gravitons of all positive masses. We need to show that these don't interfere with the validity of the usual Newtonian potential which is mediated by the zero mode.

This question has been dealt with in a general way in Ref. [1]. It is shown that the effect of the light continuum modes on physics near the centre of the brane is determined by the asymptotic behaviour of the effective potential in Eq. (5.9). Let us therefore study this behaviour.

Suppose that $\sigma \sim \mu|y|$ as $|y| \to \infty$, where μ is some constant. Integrating the condition $dz/dy = e^{\sigma}$ gives us $z \sim e^{\mu|y|}/\mu$ after choosing an arbitrary integration constant to be zero. It is then straightforward to show that as $|z| \to \infty$, Eq. (5.9) becomes[5]

$$-\frac{d^2\tilde{f}}{dz^2} + \frac{15}{4z^2}\tilde{f} = m^2\tilde{f} \tag{5.27}$$

$$\Rightarrow -\frac{d^2\tilde{f}}{dz^2} + \frac{3}{2}\left(\frac{3}{2}+1\right)\frac{1}{z^2}\tilde{f} = m^2\tilde{f}. \tag{5.28}$$

We can then appeal directly the the results in Ref. [1] to get that the gravitational potential between two masses at the centre of the domain wall is given by

$$U(r) \sim G_N \frac{M_1 M_2}{r}\left(1 + \frac{\mathcal{C}}{(kr)^2}\right), \tag{5.29}$$

where k is approximately the inverse width of the wall and $\mathcal{C}$ is some dimensionless number[6]. This is essentially identical to the results in the original Randall and Sundrum paper [10]. We conclude that if k is large enough, no deviation from the Newtonian $1/r$ potential would be detectable at scales which have so far been probed by experiment.

We have now shown that smoothed Randall-Sundrum-like models generically reproduce the familiar results of 4D gravity at low energy, including the equivalence principle, along with corrections at shorter distances.

[5] Note that the asymptotic behaviour of the effective potential is independent of μ.

[6] The constant $\mathcal{C}$ will depend on the extra-dimensional profiles of the fields corresponding to the two masses. This is because the equivalence principle won't hold for the massive graviton modes, as their profiles will not be proportional to the warp factor.

Chapter 6

Matter localisation

We have shown that reasonable 4D phenomenology can be reproduced for gravitons in a smooth Randall-Sundrum-like spacetime; now we turn to the question of scalar and fermion fields in such spacetimes. This has been studied at length in Ref. [2] by George and the present author, and this chapter is largely reproduced from that article.

The set up we will consider is exactly the same as that in chapter 5. In particular, the metric is

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{6.1}$$

where σ is a smooth function. We have seen that the effective 4D graviton spectrum corresponding to fluctuations around the metric in Eq. (6.1) consists of a single massless mode followed by a continuum of massive modes starting at $m = 0$. Contrary to naïve expectations, this does not contradict the assertion that the low-energy theory is 4 dimensional. In fact, the integrated effect of the continuum modes at the position of the brane is negligible at low energies, due to the suppression of their wavefunctions near the brane. We will now demonstrate that similar statements are true for fermions and scalar fields in the presence of such a background configuration.

We will couple fermions to the background scalar fields via the Yukawa interaction $g_j\Phi_j\overline{\Psi}\Psi$ and to gravity via the minimal coupling prescription. Including gravity in the Dirac Lagrangian requires the introduction of the vielbein V_A^N (here A is an internal Lorentz index) and the spin connection ω_N, given for the metric in Eq. (6.1) by

$$V_A^\mu = \delta_A^\mu e^\sigma \qquad \omega_\mu = \frac{i}{2}\sigma' e^{-\sigma}\gamma_\mu\gamma^5 \tag{6.2}$$

$$V_A^5 = \delta_A^5 \qquad \omega_5 = 0. \tag{6.3}$$

These yield the spin-covariant derivative $D_N = \partial_N + \omega_N$, and curved space gamma matrices $\Gamma^N = V_A^N\Gamma^A$, so that the fermion action is,

$$\mathcal{S}_\Psi = \int d^4x \int dy\, \sqrt{g}\big\{i\overline{\Psi}\Gamma^N D_N\Psi - g_j\Phi_j\overline{\Psi}\Psi\big\}. \tag{6.4}$$

where $\Gamma^\mu = \gamma^\mu, \Gamma^5 = -i\gamma^5$ with $\gamma^{\mu,5}$ the usual 4D Dirac matrices and chirality operator, respectively. The action of the $\mathbb{Z}_2$ symmetry is extended to include $y \to -y$ and $\Psi \to \Gamma^5\Psi$. For simplicity we have also imposed a global $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$, to forbid a term $g_j'\Phi_j\overline{\Psi}\Psi^c + h.c.$. The resulting Dirac equation is,

$$\left[\gamma^5\partial_y + ie^\sigma\gamma^\mu\partial_\mu - 2\sigma'\gamma^5 - g_j\phi_j(y)\right]\Psi(x^\mu, y) = 0. \tag{6.5}$$

Due to the association of γ^5 with the extra dimension, we keep track of 4D chirality when we Fourier expand Ψ. Define

$$\Psi(x,y) = \sum_n (f_L^n(y)\psi_L^n(x) + f_R^n(y)\psi_R^n(x)) \tag{6.6}$$

where ψ_L^n and ψ_R^n are respectively left- and right-handed 4D spinors satisfying the Dirac equation:

$$i\gamma^\mu\partial_\mu\psi_L^n = m_n\psi_R^n \quad i\gamma^\mu\partial_\mu\psi_R^n = m_n\psi_L^n. \tag{6.7}$$

In the massless case, these equations decouple. For $g_j\Phi_j^{\min} > 2\mu$, there is no normalisable solution for f_R^0, whereas f_L^0 is given by

$$f_L^0(y) \propto \exp\left(-\int^y ds[g_j\phi_j(s) - 2\sigma'(s)]\right). \tag{6.8}$$

This result has been obtained by a number of authors (see eg. [18, 21]). For $m_n > 0$ we get the following equations (where prime denotes differentiation with respect to y),

$$-{f_{L,R}^n}'' + 5\sigma'{f_{L,R}^n}' + \left[2\sigma'' - 6\sigma'^2 + \tilde{W}_\mp\right]f_{L,R}^n = m_n^2 e^{2\sigma} f_{L,R}^n, \tag{6.9}$$

where $\tilde{W}_\pm = (g_j\phi_j)^2 \pm g_j\phi_j' \mp g_j\phi_j\sigma'$. Notice that if we ignored gravity ($\sigma \equiv 0$), this would simply be a Schrödinger equation, and we could apply our knowledge of 1D quantum mechanics. It is in fact possible to transform Eq. (6.9) into a Schrödinger equation by changing variables. Specifically, we let $f^n_{L,R} = e^{2\sigma}\tilde{f}^n_{L,R}$, and change coordinates to $z(y)$ such that $\frac{dz}{dy} = e^\sigma$ (this is in fact a change to 'conformal coordinates', in which $ds^2 = e^{-2\sigma(y(z))}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$). Eq. (6.9) becomes,

$$\left[-\frac{d^2}{dz^2} + e^{-2\sigma}\tilde{W}_\mp\right]\tilde{f}^n_{L,R} = m_n^2\tilde{f}^n_{L,R}. \tag{6.10}$$

We thus identify the effective potential,

$$\tilde{W}^{\text{eff}}_\pm = e^{-2\sigma}\tilde{W}_\pm. \tag{6.11}$$

As $|y| \to \infty$, $\sigma \sim \mu|y|$, which in terms of z becomes $e^{-2\sigma} \sim 1/(\mu z)^2$ as $|z| \to \infty$. As $|z| \to \infty$, we have $\tilde{W}_\pm \to$ constant, and therefore the effective potential decays towards zero at large distances from the brane (see Fig. 6.1 for some specific cases). Indeed, it is an example of a 'volcano potential', familiar from analysis of the graviton sector [15]. Particles subjected to $\tilde{W}^{\text{eff}}_\pm$ are essentially free asymptotically, so there is a continuum of delta-function normalisable solutions for all $m_n^2 > 0$. We will now show that normalisability of $\tilde{f}^n_{L,R}$ implies appropriate normalisability of $f^n_{L,R}$, and conclude that our dimensionally reduced theory contains a continuum of fermions starting at zero mass.

The $\tilde{f}^n_L$ satisfy an ordinary Schrödinger equation with a continuum of eigenvalues, and therefore are delta-function orthonormalisable,

$$\int_{-\infty}^{\infty} dz\, \tilde{f}^{n*}_L \tilde{f}^{n'}_L = \delta(n - n'). \tag{6.12}$$

On the other hand, the normalisation condition for the $f^n_{L,R}$ can be derived by demanding that integrating the action in Eq. (6.4) over y leads to a properly normalised 4D kinetic term, viz.

$$\int d^4x \int dy \sqrt{g}\left\{i\overline{\Psi}\Gamma^A V^\mu_A \partial_\mu \Psi\right\} \tag{6.13}$$

$$= \int d^4x \int dn\, i\,\overline{\psi}^n \gamma^\mu \partial_\mu \psi^n. \tag{6.14}$$

Substituting in the expressions for the vielbein and metric, this condition becomes,

$$\int_{-\infty}^{\infty} dy\, e^{-3\sigma} f_L^{n*} f_L^{n'} = \delta(n - n'), \tag{6.15}$$

and similarly for the right-handed components (overlap integrals between right and left handed components don't enter). However, if we write f_L^n in terms of $\tilde{f}_L^n$ and use $dy = e^{-\sigma} dz$, we see that,

$$\int_{-\infty}^{\infty} dy\, e^{-3\sigma} f_L^{n*} f_L^{n'} = \int_{-\infty}^{\infty} dz\, \tilde{f}_L^{n*} \tilde{f}_L^{n'}. \tag{6.16}$$

So the normalisation integral for the f_L^n is equivalent to the normalisation integral for the $\tilde{f}_L^n$. Therefore there is a continuum of normalisable fermion modes in the theory, starting at zero 4D mass. Despite this, the zero modes still form an effective 4D theory at low energies. We can understand this as follows.

Flat space corresponds to $\sigma \equiv 0$, and we have seen that in this case the low-energy spectrum consists of a finite number of particles with discrete masses. The reason for this is that the effective potential of the analogue Schrödinger system asymptotes to a non-zero value; the discrete spectrum corresponds to modes bound in the potential well near $y = 0$.

Suppose now that σ is non-zero, but grows only very slowly with $|y|$. In this case, the effective potential only decays towards zero far from the brane. We thus have a localised non-zero potential in the form of a narrow well flanked by wide barriers. The low-energy eigenfunctions of such a system will generically have very small amplitudes at the position of the well, due to the potential barrier which they must tunnel through.

So although arbitrarily light fermions will exist in the theory, their wavefunctions will be strongly suppressed at the position of the brane, where the zero modes reside. They are effectively 'localised at infinity'. This leads to a very small probability of these low-energy continuum modes interacting with the zero modes.

There is one more generic feature which we expect to occur. Certain discrete energies will resonate with the potential, and the corresponding states will thus have a much larger probability of being found on the brane. These are the remnants of the discrete bound states in the flat space case, and become coincident with them in the zero-gravity limit. What happens if one of these resonant modes is produced in a high-energy process on the brane? Any particle produced on the brane will have a wavefunction truly localised to the brane, and thus cannot correspond exactly to a single mode ψ^n, which has a wavefunction oscillatory as $z \to \infty$. Instead it will be a wavepacket made from the continuum modes, with a Fourier spectrum peaked around one of the resonances. Therefore it is not a true energy/mass eigenstate, and as the various components become out of phase, the wavefunction will leak off the brane. The particle then has some probability of escaping the brane, which justifies the moniker "quasi-stable" or "quasi-localised" for the resonant modes. It is these resonant quasi-localised states that are investigated in [45].

Quantitative calculations confirming these conclusions will be given for one particular model in section 6.1.

As well as fermions, it is desirable for model building purposes to be able to localise scalar fields to the wall. In flat space, the results are similar to those for fermions, except that the mass-squared of the lightest mode depends on parameters in the 5D theory [13] (whereas in the fermion case, it is always zero). It can even be arranged to be negative, so as to realise the Higgs mechanism in the low-energy theory. We will now examine the effects of gravity on these results.

We consider a scalar field Ξ described by the action,

$$\mathcal{S}_\Xi = \int d^4x \int dy \sqrt{g} \left\{ g^{MN} (\partial_M \Xi)^\dagger \partial_N \Xi - H(\Phi, \Xi) \right\}, \tag{6.17}$$

where H describes the coupling of Ξ to itself and to the domain-wall. The linearised equation of motion for Ξ is given by,

$$\partial_M \left(\sqrt{g} g^{MN} \partial_N \Xi \right) + \sqrt{g} U(\phi_j) \Xi = 0, \tag{6.18}$$

where U is independent of Ξ, and defined by $\frac{\partial H}{\partial \Xi^\dagger} = U\Xi + \mathcal{O}(\Xi^2)$. We solve this exactly the same way as in the fermion case:

$$\text{Let} \quad \Xi(x^\mu, y) = \sum_n h^n(y) \xi^n(x^\mu), \tag{6.19}$$

where each ξ^n satisfies a 4D Klein-Gordon equation,

$$\partial^\mu \partial_\mu \xi^n + m_n^2 \xi^n = 0. \tag{6.20}$$

The analogue of Eq. (6.9) is then,

$$-h^{n\prime\prime} + 4\sigma' h^{n\prime} + U h^n = e^{2\sigma} m_n^2 h^n \tag{6.21}$$

We can convert this to a Schrödinger equation by again going to the conformal coordinate z, as well as making the substitution $h^n = e^{\frac{3}{2}\sigma}\tilde{h}^n$. This yields,

$$-\frac{d^2\tilde{h}^n}{dz^2} + \left[-\frac{3}{2}\frac{d^2\sigma}{dz^2} + \frac{9}{4}\left(\frac{d\sigma}{dz}\right)^2 + e^{-2\sigma}U\right]\tilde{h}^n = m_n^2 \tilde{h}^n. \tag{6.22}$$

As $|z| \to \infty$, $\sigma \sim \log|z|$, and $U \to$ *constant*, so we can see immediately that, as in the fermion case, the effective potential decays towards zero far from the brane. Therefore the low-energy scalar spectrum also contains a continuum of modes of arbitrarily small mass, which are properly normalisable, as can be shown by a calculation analogous to that described above for the fermions.

If $U \equiv 0$, the above equation is in fact identical to that satisfied by 4D gravitons in the background of Eq. (7.10) (see eg. Ref [1]). In this case then, we know that there is a single zero mode, followed by a continuum of modes starting arbitrarily close to $m_n^2 = 0$. The low lying continuum modes are strongly suppressed on the brane; for example, their contribution to a static potential generated by Ξ exchange between two sources on the brane separated by r, is suppressed by $1/(\mu r)^2$ relative to the contribution of the zero mode.

For non-zero U, the spectrum is modified from the graviton case, the significant difference being the possible introduction of resonant modes (in the absence of fine-tuning of parameters, there will no longer be a zero mode). As in the fermion case, these resonant modes correspond to the discrete bound modes in the corresponding gravity-free theory and we expect the first of these modes to occur for $m_n \sim \mu$. Unlike the fermion case, if appropriate coupling to the domain-wall is included, such that U makes some negative contribution to the effective potential, then there may be bound state solutions with $m_n^2 < 0$, as in the gravity-free case [13]. This signals an instability in the system, and implies that Ξ is non-zero in the stable background configuration. In this case we would have to instead solve the coupled Einstein and Klein-Gordon equations including the Φ_j fields *and* Ξ. This setup can be used for interesting model building, in which a symmetry is broken on the brane but restored in the bulk. This idea has been used in the flat space case in Ref. [25]. We sketch the reasoning following Ref. [26]. Take the scalar potential

$$H(\Phi, \Xi) = (g'\Phi^2 - u^2)\Xi^\dagger\Xi + \tau(\Xi^\dagger\Xi)^2 \tag{6.23}$$

where we have specialised to a single background field Φ, and we assume $g'\Phi^{\min} - u^2 > 0$ such that $(\Phi, \Xi) = (\pm\Phi^{\min}, 0)$ are still the global minima of the potential, and we must have $\Xi \to 0$ as $|y| \to \infty$. If Φ forms a domain wall, then $\Phi \sim 0$ inside the wall, so that the leading term of $H(\Phi, \Xi)$ is $\sim -u^2\Xi^\dagger\Xi$, suggesting that the $\Xi = 0$ solution is unstable there. This will show up as a negative eigenvalue $m_n^2 < 0$ in equation (6.22), and solving for a consistent set of background solutions will yield a background Ξ that is peaked on the brane and tending to zero in the bulk. Putting Ξ in a non-trivial representation of some gauge group will induce spontaneous breaking of that group on the brane, a mechanism which can be used, for example, to realise the standard model Higgs mechanism on the brane.

In the stable case then, asymptotically U will approach some constant positive value U_0. As $|z| \to \infty$, we can approximate the effective potential V_{eff} as follows:

$$V_{\text{eff}} \sim \frac{1}{z^2}\left(\frac{15}{4} + \frac{U_0}{\mu^2}\right). \tag{6.24}$$

Again we can appeal to the results of Ref. [1], where it is shown that for a potential that behaves asymptotically as $\alpha(\alpha+1)/z^2$, the amplitudes of modes with small m_n are suppressed by $(m_n/\mu)^{\alpha-1}$. Therefore the coupling to the domain-wall actually reduces the effect of the continuum modes on low energy physics.

6.1 Toy model calculation

The above conclusions can be illustrated concretely by finding numerical results for a specific case. We will make use of a background solution found in Ref. [16]

in which a single real scalar field forms the domain-wall. The stability of this solution is demonstrated in Ref. [16]. The solutions for the warp factor and scalar field are[1],

$$\begin{aligned}\sigma(y) &= a\log(\cosh(ky)) \\ \phi(y) &= D\arctan(\sinh(ky)),\end{aligned} \tag{6.25}$$

where $a = D^2/12M^3$ is proportional to the 5D Newton's constant, and the solution corresponds to a 5D cosmological constant given by $\Lambda = -D^4k^2/6M^3$. The scalar field potential which admits this solution is,

$$V(\Phi) = 6ak^2M^3(1+4a)\sin^2\left(\frac{\phi}{D} - \frac{\pi}{2}\right). \tag{6.26}$$

We'd like to study a fermion field Ψ in the above background to illustrate the existence and suppression of the low-lying continuum modes. We again impose a global $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$ so that Ψ only couples to the background via a term $g\Phi\overline{\Psi}\Psi$. For the sake of examining interactions later, we also include a scalar field Ξ, which $U(1)$ acts on via $\Xi \to e^{2i\theta}$, to mediate interactions between Ψ quanta[2], and take the action describing these two fields to be

$$\begin{aligned}\mathcal{S}_{\Psi\Xi} = \int d^4x \int dy\sqrt{G}\Big[i\overline{\Psi}\Gamma^M\partial_M\Psi - g\Phi\overline{\Psi}\Psi + \partial^M\Xi^\dagger\partial_M\Xi - g'\Phi^2\Xi^\dagger\Xi - u^2\,\Xi^\dagger\Xi \\ - \tau(\Xi^\dagger\Xi)^2 - \lambda(\Xi\overline{\Psi}\Psi^c + \text{h.c.})\Big],\end{aligned} \tag{6.27}$$

where $\Psi^c = \Gamma^2\Gamma^5\Psi^*$. Note that the full action includes the background given by Eq. (5.1). We have given Ξ a non-trivial potential to induce resonant modes, which can interact strongly on the brane. For our background solution to remain stable, $\Xi = 0$ must be the stable solution i.e. Eq. (6.22) must not have any negative eigenvalues. Choosing $u^2 > 0$ suffices to guarantee this.

The effective Schrödinger equation for the fermion field is found as described earlier. For various values of a, the resulting effective potential felt by the left

[1]The functional form of this solution appears different to that given in Ref. [16], but it is in fact equivalent.

[2]It is necessary to introduce a field other than Φ, because the fermion zero mode is chiral, and thus does not interact with Φ via the term $g\Phi\overline{\Psi}\Psi$. Additionally, while the modes of Φ mix with scalar gravitational degrees of freedom, the global $U(1)$ symmetry prevents such a mixing of Ξ modes.

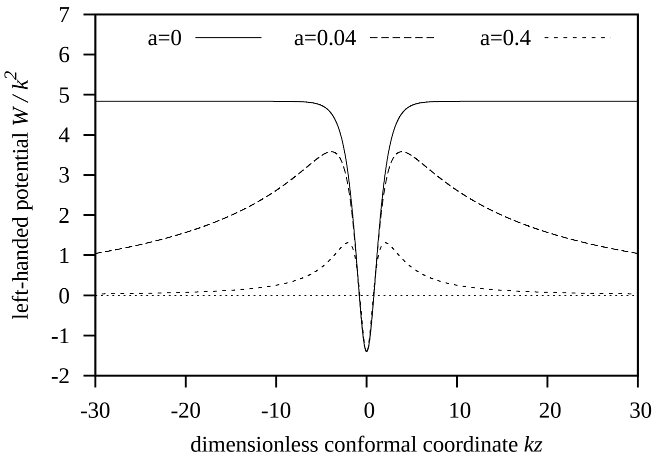

Figure 6.1: An example of the effective Schrödinger potential, $\tilde{W}_-^{\text{eff}}$, which traps a left-handed fermion field. The three graphs correspond to no gravity ($a = 0$), 'weak' gravity ($a = 0.04$), and 'strong' gravity ($a = 0.4$). The horizontal line is $W = 0$. All plots have $gD = 1.4\,k$.

chiral component of the fermion is plotted in Fig. 6.1. It does indeed asymptote to zero when gravity is included, implying the existence of a continuum of arbitrarily light modes. There is of course a zero mode which is localised to the brane – all other modes are however oscillatory at infinity.

We wish to quantify our argument that the light continuum modes do not overly influence physics on the brane. The dominant process by which they could be detected in our toy model is two zero mode particles annihilating to produce two continuum particles via exchange of a Ξ quantum, so this is the process we will consider. As explained earlier (for fermions, but the physics is the same), a Ξ quantum produced on the brane will not correspond to a single mass mode, but will be a wavepacket initially localised on the brane. The creation of such a wavepacket on the brane and the ensuing shape of the wavepacket will be a complicated issue, and is not considered here. Instead we will simply take $\text{sech}(kz)$ as a typical localised profile[3] and assume that a Ξ quantum is produced with the extra-dimensional wavefunction $\tilde{h}(z) = \sqrt{k/2}\,\text{sech}(kz)$. We

[3]Results should be almost identical for any profile which decays exponentially beyond $|z| \sim 1/k$.

have computed the Fourier decomposition of $\tilde{h}(z)$ in terms of the mass eigenmodes (the eigenfunctions of Eq. (6.22)); the spectrum is sharply peaked at a mass corresponding to the first resonant mode, as expected. We now proceed to calculate the effective coupling of the fermion modes to this particle in the dimensionally reduced theory. This will give us a quantitative estimate of the likelihood of continuum fermion modes being produced by on-brane dynamics through s-channel annihilation. It will also be a valid estimate for t-channel scattering of localised zero modes with bulk continuum modes.

The effective coupling constant between the fermion modes of Ψ and the localised Ξ particle is given by the 5D Yukawa coupling constant multiplied by the overlap integral of their extra-dimensional wavefunctions. For the fermion mode with extra-dimensional dependence $f^n(y)$, the coupling will be

$$\begin{aligned} \lambda_n^{(4)} &= \lambda \int dy\, e^{-4\sigma}\, h(y)\, \big(f^n(y)\big)^2 \\ &= \lambda \int dz\, e^{\frac{1}{2}\sigma} \tilde{h}(z) \big(\tilde{f}^n(z)\big)^2 \\ &= \lambda \sqrt{\frac{k}{2}} \int dz\, e^{\frac{1}{2}\sigma} \frac{\big(\tilde{f}^n(z)\big)^2}{\cosh kz}. \end{aligned} \tag{6.28}$$

The results for the case $a = 0.04$ are plotted in Fig 6.2, contrasted with the results in the gravity-free case[4]. It is clear that the 4D coupling constants go quickly to zero for modes with masses much less than the inverse width k of the domain-wall.

Such behaviour is of course easy to understand, based on the earlier discussion of light continuum modes. Modes with energy much less than k see a wide potential barrier preventing them from penetrating to the brane, where the Ξ particle resides. At an energy approximately equal to k, we see the first resonant mode, which does not suffer the generic suppression near the brane. Continuum modes with energy above the barrier height ($\sim 5k^2$ for the gravity-free case and $\sim 3.5k^2$ for weak gravity) are free to roam in the vicinity of the brane, hence their coupling is of order unity. We have explicitly plotted the profiles of a

[4]Note that what is plotted is really "interaction strength per continuum mode" with mode energy used on the horizontal axis to label a particular mode number. An integral over some finite range of modes is required to yield a finite on-brane effect.

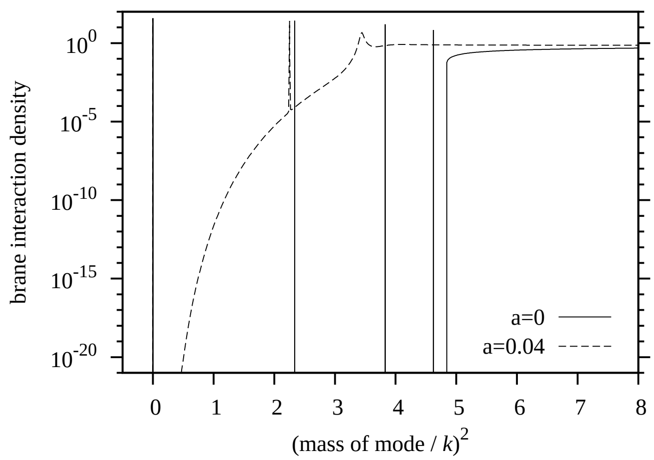

Figure 6.2: The "interaction per continuum mode", $\lambda_n^{(4)}/\lambda\sqrt{k}$, for continuum modes interacting with a typical bound mode on the brane. Both the gravity-free ($a = 0$) and 'weak' gravity ($a = 0.04$) cases are shown. The fermion zero mode remains bound in the presence of gravity (hidden by the gravity-free plot), whilst a continuum is introduced for all positive values of the mass. It is clear that at energies well below k, the continuum modes are essentially decoupled from those on the brane. The coupling becomes relatively strong for energies greater than the maximum of the effective potential. The fermion coupling strength is $gD = 1.4\,k$.

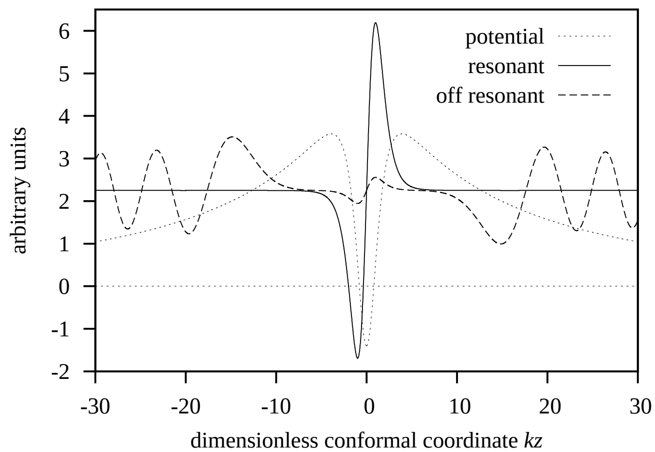

Figure 6.3: The extra-dimensional profiles of the resonant mode at $(E/k)^2 \simeq 2.3$, and a mode off-resonance by 2.0×10^{-4} in units of $(E/k)^2$. We are in the 'weak' gravity case with $a = 0.04$. The profiles are not plotted on the same scale; in reality, each is normalised to the same amplitude *at infinity* (since the normalisation condition is dominated by the behaviour of the wavefunction at infinity). Thus the contrast is much more dramatic even than it appears here.

resonant mode and a (slightly) off-resonant mode in Fig. 6.3, to illustrate the amplification of one and suppression of the other on the brane.

Part III

An $SU(5)$ GUT in 4+1 non-compact dimensions

Chapter 7

The scalar + gravity background

We have somewhat anticipated our final model by studying gravitating domain walls in previous sections, but let's now outline the basic idea which we will pursue.

We wish to start with a theory which treats all dimensions equally, so on this basis we reject the concept of compact extra dimensions. The only other option known for reproducing 4D gravity is the RS2 model and its generalisations. Because we are looking for a field theory model, we will not use the infinitely-thin brane of the RS2 model, but instead appeal to a gravitating scalar field domain wall as our background. This will spontaneously break the spacetime symmetry down to 4D Poincaré symmetry, as well as localising gravity.

For the matter sector, we have seen that 4D scalars and chiral fermions can both also be localised via coupling to the domain wall. This leaves us just the gauge bosons to localise. The only mechanism we have seen which localises gauge bosons in a way which preserves gauge universality is that of Dvali and Shifman, so that is what we will utilise.

If we wish to appeal to the Dvali-Shifman mechanism to localise standard model gauge bosons, we must start with a larger gauge group in the higher-dimensional theory. The obvious choice is the minimal grand unification group $SU(5)$.

In the standard $SU(5)$ theory, spontaneous breaking to the standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is achieved via a scalar field in the adjoint, or **24** representation. To implement the Dvali-Shifman mechanism therefore, we will need a real scalar singlet η, as well as a scalar field in the **24** representation, which we will call χ.

We can write $\chi = \chi^a t^a$, where $\{t^a\}$ is the set of generators for $SU(5)$, and $U \in SU(5)$ acts on χ via $\chi \to U\chi U^\dagger$. The normalisation is such that $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.

Because we will be looking for a domain wall solution, we want the scalar potential to have a doubly degenerate minimum. Thus we impose a $\mathbb{Z}_2$ symmetry on our theory, under which $\eta \to -\eta$ and $\chi \to -\chi$. The most general quartic potential invariant under $SU(5) \times \mathbb{Z}_2$ is then[1]

$$V(\eta, \chi) = (c\eta^2 - \mu_\chi^2)Tr(\chi^2) + \lambda_1 \left[Tr(\chi^2)\right]^2 + \lambda_2 Tr(\chi^4) + l(\eta^2 - v^2)^2. \quad (7.1)$$

We need to impose additional constraints on the parameters to guarantee that $(\eta, \chi) = (\pm v, 0)$ corresponds to the global minimum of the potential, allowing for stable domain wall solutions of the form we require. The detailed potential analysis is carried out in Appendix A, where it is shown that the following inequalities suffice:

$$cv^2 - \mu_\chi^2 > 0, \quad 4l\tilde{\lambda}v^4 - \mu_\chi^4 > 0, \quad (7.2)$$

where $\tilde{\lambda} \equiv \lambda_1 + \frac{7}{30}\lambda_2$.

To break $SU(5)$ to the standard model gauge group, χ must develop an expectation value in the direction of the hypercharge generator Y, which in $SU(5)$ space is given by

$$Y = \frac{1}{2\sqrt{15}}\mathrm{diag}(2, 2, 2, -3, -3). \quad (7.3)$$

Therefore we will look for a solution in which only this component of χ is non-zero. For convenience we redefine χ to refer to just the component in the direction of Y.

[1] In fact, we are leaving out a term proportional to $\eta\mathrm{Tr}(\chi^3)$ for reasons of simplicity

We will assume that the solution depends only on the extra-dimensional coordinate y, and let prime denote differentiation with respect to y. The Klein-Gordon equations which follow from the potential of Eq (7.1) are then

$$\eta'' - \eta \left\{4l(\eta^2 - v^2) + c\chi^2\right\} \quad = \quad 0 \tag{7.4}$$

$$\chi'' - \chi \left\{\tilde{\lambda}\chi^2 + c\eta^2 - \mu_\chi^2\right\} \quad = \quad 0, \tag{7.5}$$

We take the simplest ansatz which satisfies the criteria for the Dvali-Shifman mechanism:

$$\begin{aligned} \eta(y) = &\quad v\tanh(ky) \\ \chi(y) = &\quad \frac{A}{\cosh(ky)}, \end{aligned} \tag{7.6}$$

where k and A are positive constants to be chosen so that the equations of motion are satisfied. If we impose the following parameter condition[2],

$$2\mu_\chi^2\left(\frac{c}{\tilde{\lambda}} - 1\right) + \left(2c - 4l - \frac{c^2}{\tilde{\lambda}^2}\right)v^2 = 0, \tag{7.7}$$

then we obtain the solution

$$k^2 \quad = \quad cv^2 - \mu_\chi^2 \tag{7.8}$$

$$A^2 \quad = \quad \frac{1}{\tilde{\lambda}}(2\mu_\chi^2 - cv^2). \tag{7.9}$$

We expect this solution to be stable, as we have demonstrated the stability of many similar numerical solutions using the techniques described in Ref. [3].

It is promising that we can find nice solutions of this form, but if we hope to construct a realistic theory, we cannot neglect gravity as we have above. To consistently incorporate gravity, we need to find a background solution for the scalar fields and metric together. We can then study the motion of other fields in this background.

We are looking for a smoothed-out version of the Randall-Sundrum metric, so we take as an ansatz,

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{7.10}$$

[2]This is only to obtain the analytic solution given. Numerical solutions with the same features can be obtained for any values of the parameters satisfying the inequalities (7.2).

and correspondingly assume that the solution for the scalar fields depends only on y. The equations of motion are the Einstein equations for the metric with the stress-energy tensor of the scalar fields as the source, and the Klein-Gordon equation in the curved background (7.10). Explicitly, Einstein's equations reduce to

$$\sigma'' = \frac{1}{6M^3}\left\{\frac{1}{2}\left(\frac{d\eta}{dy}\right)^2 + Tr\left[\left(\frac{d\chi}{dy}\right)^2\right]\right\} \tag{7.11}$$

$$(\sigma')^2 = \frac{1}{24M^3}\left\{-\Lambda + \frac{1}{2}\left(\frac{d\eta}{dy}\right)^2 + Tr\left[\left(\frac{d\chi}{dy}\right)^2\right] - V(\eta,\chi)\right\}, \tag{7.12}$$

while the Klein-Gordon equations for η and χ are,

$$\frac{d^2\eta}{dy^2} - 4\sigma'\frac{d\eta}{dy} = \frac{\partial V}{\partial \eta} \tag{7.13}$$

$$\frac{d^2\chi_j}{dy^2} - 4\sigma'\frac{d\chi_j}{dy} = \frac{\partial V}{\partial \chi_j} \tag{7.14}$$

Before we begin solving the system, we note something important. If we differentiate the second Einstein equation (7.12), the resulting equation is satisfied identically if we assume that the first Einstein equation (7.11), as well as the two Klein-Gordon equations (7.13, 7.14), are satisfied. Therefore if we solve equations (7.11, 7.13, 7.14), equation (7.12) simply tells us the value of the 5D cosmological constant Λ.

We will take an ansatz that identically satisfies (7.11),

$$\eta = v\tanh(ky) \qquad\qquad \chi = \frac{v}{\cosh(ky)} \tag{7.15}$$

$$\sigma = \tfrac{a}{2}\log\left[\cosh(ky)\right] \tag{7.16}$$

where k is some positive constant to be determined, and $a \equiv \frac{v^2}{6M^3}$ (notice that a is proportional to the 5D Newton's constant, so measures the strength of gravity). Plugging these expressions into the Klein-Gordon equations, we find that we get a solution as long as the following relations are satisfied,

$$c = 2l + \frac{\tilde{\lambda}}{2} \tag{7.17}$$

$$\mu_\chi^2 = \frac{lv^2}{1+a} + \frac{\tilde{\lambda}}{4}\left(\frac{3+4a}{1+a}\right). \tag{7.18}$$

It is shown in Appendix A that these relations do not contradict the necessary inequalities (7.2). The value of k is given by

$$k^2 = \frac{cv^2 - \mu_\chi^2}{1+2a}. \tag{7.19}$$

The final quantity we need to calculate is the cosmological constant Λ. As mentioned above, this follows directly from Eq. (7.12). It is however not necessary to put in the full form of our solution; because Λ is constant, we can simply evaluate it at a convenient value of y. Inspecting our solution, we can see that it will be easiest to evaluate in the limit $y \to \infty$. In this limit we have $\sigma' \to ak/2$ and $d\eta/dy, d\chi/dy, V \to 0$, so some simple algebra gives

$$\Lambda = -ak^2v^2. \tag{7.20}$$

This is an example of the general fact that Randall-Sundrum-like spacetimes necessarily have a negative bulk cosmological constant.

Chapter 8

Completing the model, and initial analysis

We have found a background solution which we claim will localise the Standard Model gauge bosons to the domain wall. We now need to add matter fields to the 5D theory in such a way as to reproduce the standard model spectrum at low energies. The following is presented in brief in Ref. [4].

8.1 The matter content

In the usual 4D SU(5) GUT, one family of Standard Model fermions fits into two representations, the **10** and $\mathbf{5}^*$. We will concentrate for now on one family, and therefore include in our model 5D fermions Ψ_5 and Ψ_{10} in these two representations. These are best represented as a 5-vector and an anti-symmetric 5×5 matrix respectively, transforming under SU(5) as $\Psi_5 \to U^* \Psi_5$ and $\Psi_{10} \to U \Psi_{10} U^T$.

We also include a scalar field Φ in the $\mathbf{5}^*$, for this contains an SU(2) doublet which can hopefully be used to realise electroweak symmetry breaking.

To write down the action for our model, we need first to decide how the $\mathbb{Z}_2$ symmetry acts on these matter fields. We will restrict terms in our action to those operators of lowest mass dimension which nonetheless yield non-trivial interactions. Therefore the only terms which can couple our fermions to the scalar field background are standard Yukawa couplings such as $\eta\overline{\Psi}_n\Psi_n$. Both our scalar fields change sign under $\mathbb{Z}_2$, so for these terms to respect the symmetry, the fermion bilinear $\overline{\Psi}_n\Psi_n$ must also change sign. This can be achieved by $\Psi_n \to i\Gamma^5\Psi_n$. To ensure that the fermion kinetic terms remain invariant, we also define the discrete symmetry to include a reflection in the extra dimension: $y \to -y$.

The fermion mass terms that arise from electroweak symmetry breaking can be incorporated into the SU(5) theory through the terms $\overline{(\Psi_5^c)}\Psi_{10}\Phi$ and $\epsilon^{ijklm}\overline{(\Psi_{10}^c)}_{ij}\Psi_{10kl}\Phi^*_m$. It can be checked that these fermion bilinears don't change sign under $\mathbb{Z}_2$, so these terms can be included in our 5D theory if Φ is a singlet under $\mathbb{Z}_2$. The Yukawa Lagrangian consistent with the symmetries of the theory is then

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} = &-\big(h_{5\eta}\overline{\Psi}_5\Psi_5\eta + h_{5\chi}\overline{\Psi}_5\chi^T\Psi_5 + h_{10\eta}Tr(\overline{\Psi}_{10}\Psi_{10})\eta \\ &- 2h_{10\chi}\text{Tr}(\overline{\Psi}_{10}\chi\Psi_{10}) + h_-\overline{(\Psi_5^c)}\Psi_{10}\Phi + h_+\epsilon_{ijklm}\overline{(\Psi_{10}^c)}_{ij}\Psi_{10kl}\Phi^*_m\big).\end{aligned} \tag{8.1}$$

We will split the potential for the scalars into two pieces, one involving just η and χ, which will form our background domain wall, and the other containing all the terms depending on Φ. We write $V = V_{\eta\chi} + V_{\text{rest}}$, where

$$V_{\eta\chi} = (c\eta^2 - \mu_\chi^2)\text{Tr}(\chi^2) + \lambda_1\left[\text{Tr}(\chi^2)\right]^2 + \lambda_2\text{Tr}(\chi^4) + l(\eta^2 - v^2)^2, \tag{8.2}$$

and

$$\begin{aligned}V_{\text{rest}} = &\ \mu_\Phi^2\Phi^\dagger\Phi + \lambda_3(\Phi^\dagger\Phi)^2 + \lambda_4\Phi^\dagger\Phi\eta^2 + 2\lambda_5\Phi^\dagger\Phi\text{Tr}(\chi^2) \\ &+ \lambda_6\Phi^\dagger(\chi^T)^2\Phi + \lambda_7\Phi^\dagger\chi^T\Phi\eta.\end{aligned}$$

The action for our theory is then

$$\mathcal{S} = \int d^4x \int dy(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Yuk}} - V), \tag{8.3}$$

where $\mathcal{L}_{\text{kin}}$ contains all the SU(5) gauge covariant kinetic terms. The theory is invariant under SU(5) gauge symmetry, as well as the discrete $\mathbb{Z}_2$ symmetry $\chi \to -\chi, \eta \to -\eta, y \to -y$ and $\Psi_n \to i\Gamma^5\Psi_n$.

8.2 Localisation profiles without gravity

We are relying on the non-perturbative SU(5) dynamics to localise the standard model gauge bosons to the domain wall (Dvali-Shifman mechanism). This will also affect the fermions, because to propagate off the wall, fermions will have to be incorporated into (massive) SU(5) hadrons. However, close to the wall we expect these non-perturbative effects to be negligible due to the breaking of SU(5) there. Therefore we assume that we can learn something of the structure of the theory by studying the classical localisation profiles of the matter fields. To get a feel for the general features of the model without worrying about the complications due to gravity, we first work in the background of the flat space solution in Eq. (7.6), which we repeat here for convenience:

$$\begin{aligned} \eta(y) = & \quad v \tanh(ky) \\ \chi(y) = & \quad \frac{A}{\cosh(ky)}, \end{aligned} \tag{8.4}$$

where χ denotes just the component of χ proportional to the hypercharge generator, and

$$k^2 \quad = \quad cv^2 - \mu_\chi^2 \tag{8.5}$$

$$A^2 \quad = \quad \frac{1}{\tilde{\lambda}}(2\mu_\chi^2 - cv^2). \tag{8.6}$$

8.2.1 Fermions

First we notice something interesting about the way the fermions couple to the background scalar fields. Because χ is in the adjoint representation of the gauge group, it will couple to the fermions in the same way as do the gauge bosons. This means that because the vev of χ is in the direction of the hypercharge generator, the potential felt by the fermions will depend on their hypercharge. The resulting Dirac equations thus depend on both the hypercharge of the fermion and which multiplet it belongs to. We can write all the equations together as

$$\left[i\Gamma^A \partial_A - h_{n\eta}\eta(y) - \sqrt{\frac{3}{5}}\frac{Y}{2} h_{n\chi}\chi(y) \right] \Psi_{nY}(x,y) = 0. \tag{8.7}$$

Assuming then that $h_{n\eta} > 0$, each fermion species will have a localised left-handed zero mode. Finding the extra-dimensional profiles proceeds as per chapter 6, and yields

$$f^0_{LY} \propto \exp\left\{ -\int dy \left(h_{n\eta}\eta(y) + \sqrt{\frac{3}{5}}\frac{Y}{2} h_{n\chi}\chi(y) \right) \right\}. \tag{8.8}$$

The peak of each profile is given by the zero of the integrand in the exponent, viz.

$$\sinh ky = -\sqrt{\frac{3}{5}}\frac{Y}{2}\frac{\lambda_{n\chi}A}{\lambda_{n\eta}v}. \tag{8.9}$$

Thus the fermions will be localised at different points along the extra dimension according to their hypercharge; the split fermion idea of Ref. [31] is realised automatically. We have some choice over their distribution, because we can choose the relative sign and size of $h_{5\chi}$ and $h_{10\chi}$. The most important consideration in making this decision is stability of the proton. The most significant operators that mediate proton decay in the SU(5) theory are $\overline{u}_R(e_R)^c\phi_c^*$ and $\overline{d}_R(u_R)^c\phi_c$ together, where ϕ_c is the colour-triplet scalar coming from the $\mathbf{5}^*$. So for suppression of proton decay it will suffice to separate d_R, which comes from Ψ_5 and u_R, which comes from Ψ_{10}. Because the hypercharges of these fields have opposite sign, we can localise them on opposite sides of the wall by choosing $h_{5\chi}$ and $h_{10\chi}$ to have the same sign. An example of the resulting profiles is shown in Fig. 8.1.

8.2.2 Higgs field

Just like the fermions, the various components of the scalar field Φ will see a potential which depends on their hypercharge. Φ contains a colour triplet with hypercharge $-1/\sqrt{15}$ and an electroweak doublet with hypercharge $3/2\sqrt{15}$. Thus we look for the coloured triplet modes $p_c^m(y)\phi_c^m(x)$ and electroweak doublet modes $p_w^m(y)\phi_w^m(x)$ separately, where $\phi_{c,w}^m$ are 4D scalar fields satisfying $\partial^\mu\partial_\mu\phi_{c,w}^m + m^2\phi_{c,w}^m = 0$. The possible values of the mass-squared m^2 are determined as the eigenvalues of an analogue Schrödinger equation, as explained in chapter 6. The corresponding potential is

$$W_Y(y) = \mu_\Phi^2 + \lambda_4\eta^2(y) + \lambda_5\chi^2(y) + \frac{3Y^2}{20}\lambda_6\chi^2(y) + \sqrt{\frac{3}{5}}\frac{Y}{2}\lambda_7\eta(y)\chi(y), \tag{8.10}$$

where Y is the hypercharge of the relevant component of Φ. The potential seen by the triplet and doublet respectively is plotted in Fig. 8.2. As suggested by

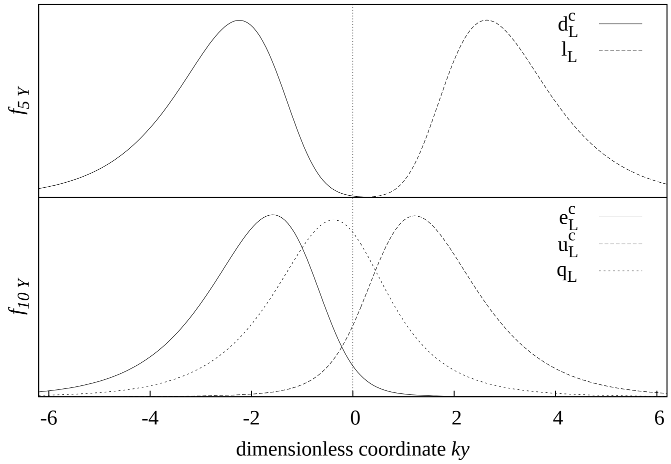

Figure 8.1: Typical extra-dimensional profiles $f_{nY}(y)$ for the fermions contained in the 5^* (top) and the 10 (bottom). The fields η and χ are as per Eq. (8.4) and parameter choices are: $v = A = 1$, $h_{n\eta} = 1$, $h_{5\chi} = 6$, $h_{10\chi} = 1$. The profiles are normalised such that $\int dy f_{nY}^2(y) = 1$.

the figure, the potential well seen by the doublet is generically deeper than that seen by the triplet. Parameters can be chosen such that there exists a bound state for the doublet with negative mass-squared, but not so for the triplet. This means that electroweak symmetry will be broken on the domain wall, but colour symmetry will not. We can see now that as well as suppressing proton decay, our theory also generically removes the unwanted prediction of the usual $SU(5)$ theory that the bare masses of the electron and the down quark are equal. This happens because although they still each have the same coupling to Φ, they have different profiles in the extra dimension, and thus different overlaps with ϕ_w. This leads to different effective couplings in the low-energy theory, and thus different masses after electroweak symmetry breaking.

8.3 Breaking down the $SU(5)$ Yukawas

The Standard Model Yukawa couplings of the fermions to the Higgs field are contained in the $SU(5)$-invariant couplings to Φ in the Lagrangian (8.1). Therefore we need to break these interactions down in terms of their Standard Model

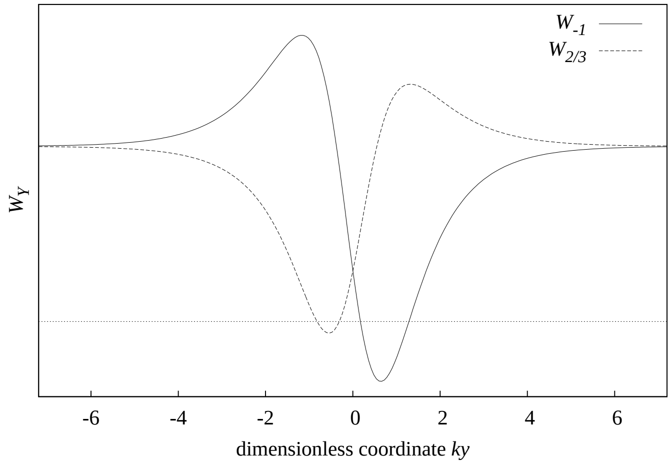

Figure 8.2: Typical effective potentials seen by the electroweak doublet and coloured triplet contained in Φ. The horizontal line is $W = 0$.

content, to identify Standard Model parameters in terms of the parameters of the full theory. To begin, we write our two fermion multiplets and one scalar multiplet schematically as

$$\Psi_5 = \begin{pmatrix} D' \\ L \end{pmatrix} \quad \Psi_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{(3)} \cdot U' & Q \\ -Q & \epsilon^{(2)} E' \end{pmatrix} \quad \Phi = \begin{pmatrix} \Phi_c \\ \Phi_w \end{pmatrix}. \tag{8.11}$$

This notation needs a little explaining. We use $\epsilon^{(n)}$ for the n-index alternating symbol, and define $(\epsilon^{(3)} \cdot U')_{ij} = \epsilon_{ijk} U'_k$. The Standard Model representations of the various fields are

$$\begin{aligned} D' \sim (3^*, 1, \frac{2}{3}) \quad L \sim (1, 2, -1) \quad U' \sim (3^*, 1, \frac{4}{3}) \\ Q \sim (3, 2, \frac{1}{3}) \quad E' \sim (1, 1, 2) \\ \Phi_c \sim (3^*, 1, \frac{2}{3}) \quad \Phi_w \sim (1, 2, -1). \end{aligned} \tag{8.12}$$

The fields D' and U' carry the quantum numbers of the Standard Model fields d^c and u^c respectively, but we avoid the notation D^c, U^c because of possible confusion with the charge conjugation operation in 5D. We will use $i, j, \ldots$ to denote $SU(3)$ indices and $\alpha, \beta, \ldots$ to denote $SU(2)$ indices. Then the Yukawa couplings of Φ become

$$\overline{\Psi^c_{5i}}\Psi_{10ij}\Phi^*_j = \frac{1}{\sqrt{2}}\big(\epsilon_{ijk}\overline{D^c_i}U'_k\Phi^*_{ci} + \overline{D^c_i}Q_{i\alpha}\Phi^*_{w\alpha} - \overline{L^c_\alpha}Q_{i\alpha}\Phi^*_{ci} + \epsilon_{\alpha\beta}\overline{L^c_\alpha}E'\Phi^*_{w\beta}\big)$$

$$\epsilon_{ijklm}\overline{(\Psi^c_{10})}_{ij}\Psi_{10kl}\Phi^*_m = 2\big(\epsilon_{\alpha\beta}\overline{U'^c_i}Q_{i\alpha}\Phi^*_{w\beta} + \overline{U'^c_i}E'\Phi^*_{ci} + \epsilon_{\alpha\beta}\overline{Q^c_{i\alpha}}U'_i\Phi^*_{w\beta} + \overline{E'^c}U'_i\Phi^*_{ci} - \epsilon_{ijk}\epsilon_{\alpha\beta}\overline{Q^c_{i\alpha}}Q_{j\beta}\Phi^*_{ck}\big). \tag{8.13}$$

This decomposition is actually fairly easy to do, by realising that because the interactions are $SU(5)$ invariant, each term in the above must be $SU(3) \times SU(2) \times U(1)$ invariant. It's then just a matter of choosing a single value for the indices in each term, to check the overall factor, and finally counting the terms to ensure nothing has been missed (25 individual terms for the first Yukawa, corresponding to 25 choices for the indices (i, j), and $120 = 5!$ for the second, corresponding to all perturbations of the indices (i, j, k, l, m)).

We can identify in the above the analogues of the usual Standard Model mass terms for the fermions, as well as interactions with the coloured scalar field Φ_c.

8.4 Including gravity

With gravity included, the action for our model is

$$\mathcal{S} = \int d^4x \int dy\, \sqrt{g}(-2M^3R - \Lambda + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Yuk}} - V), \tag{8.14}$$

where g is the magnitude of the determinant of the metric, M the 5D gravitational mass scale, R the scalar curvature, and Λ the cosmological constant. All other terms now include minimal coupling to gravity.

The background solution we will use was found in chapter 7, and is given by

$$\eta = v\tanh(ky) \qquad \chi = \frac{v}{\cosh(ky)} \qquad \sigma = \frac{a}{2}\log\left[\cosh(ky)\right] \tag{8.15}$$

where the metric is $ds^2 = e^{-2\sigma}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$, and

$$k^2 = \frac{cv^2 - \mu^2_\chi}{1 + 2a}. \tag{8.16}$$

The Dirac equation obeyed by the fermions now includes terms arising from the spin-connection, and becomes

$$\left[\gamma^5\partial_y + ie^{\sigma}\gamma^{\mu}\partial_{\mu} + 2\sigma'\gamma^5 - h_{n\eta}\eta(y) - \sqrt{\frac{3}{5}}\frac{Y}{2}h_{n\chi}\chi(y)\right]\Psi_{nY} = 0 \qquad (8.17)$$

The solutions for the zero modes are thus

$$f^0_{LY} \propto \exp\left\{-\int dy \left(h_{n\eta}\eta(y) + \sqrt{\frac{3}{5}}\frac{Y}{2}h_{n\chi}\chi(y) - 2\sigma'(y)\right)\right\}. \qquad (8.18)$$

Asymptotically, $2\sigma' \to ak$, so it can be seen that this is localised to the brane, and normalisable, as long as[1] $hv > ak$.

For the scalar field Φ, we can see from equation (6.22) that the effective potential near $y = 0$ will still be deeper for the electroweak doublet ϕ_w than for the colour triplet ϕ_c. Therefore with an appropriate choice of parameters, it will still be possible to obtain a bound mode of the electroweak doublet with a tachyonic mass, while keeping all colour triplet modes massive. This means that a truly stable background solution will have a non-zero value for ϕ_w inside the wall as well as the adjoint χ. The mass parameters associated with ϕ_w will be approximately equal to the electroweak scale, which is much smaller than the energy scales associated with the background η, χ solution. Therefore we expect the background solutions for the metric, η and χ to be affected only minimally. There is however a profound consequence for fermion localisation arising from a non-zero ϕ_w background. This is because it couples *different* fermion fields to each other, which the other scalar fields do not. Such a situation is studied in Ref. [46], where it is found that zero modes no longer exist for the fermions in this case. This is easy to understand heuristically; what were the zero modes in the case without ϕ_w are now coupled together by mass terms in the same way as in usual electroweak symmetry breaking. The arguments of chapter 6 then imply that these modes are no longer truly bound (as indeed is pointed out in Ref. [46]), and therefore have a finite probability of escaping from the brane. Parameters will have to be chosen such that the lifetimes of the Standard Model

[1] In fact the condition for normalisability is weaker than this, due to the warp factor appearing in the spacetime measure.

particles on the brane are sufficiently long to avoid contradicting the fact that matter has never been seen to disappear.

8.5 Energy scales

Our model contains a number of *a priori* unspecified energy scales: the 5D gravitational scale M, the confinement scale $\Lambda_{SU(5)}$ of the $SU(5)$ theory in the bulk, and the inverse width k of the domain wall. We do in fact require a certain hierarchy between these scales for our model, as described, to be consistent. Firstly, we don't want quantum gravity effects to interfere with the field-theoretic confinement, so require $\Lambda_{SU(5)} < M$.

In Ref [28] it is stated that for the Dvali-Shifman mechanism to work it is necessary that the massive gauge bosons on the domain wall must be heavier than $\Lambda_{SU(5)}$, and that the wall must be wider than $\Lambda^{-1}_{SU(5)}$ (the approximate size of an $SU(5)$ hadron). These conditions are supposed to suppress the effects of $SU(5)$ confinement inside the wall. We can understand this heuristically by noticing that the static potential due to a massive photon is suppressed by e^{-mr} at a distance r, and extrapolating this to all interactions with massive vector bosons. Thus if some of the $SU(5)$ gauge bosons gain masses larger than $\Lambda_{SU(5)}$, $SU(5)$ hadronisation will not occur because this requires gauge interactions over distances of $\Lambda^{-1}_{SU(5)}$. In the context of our solution of Eq. (8.15), these conditions roughly become $v^{\frac{2}{3}} > \Lambda_{SU(5)} > k$.

Finally, since k is approximately the mass of the first Kaluza-Klein excitation we would observe, this must be sufficiently large to explain the fact that we have not made any such observations. We note that deviations from Newton's law occur at a distance of $\sim 1/k$; the current measurements at sub-millimetre scales give only $k >$ 10eV. Therefore the best bounds on k will certainly come from particle physics experiments, and most likely from precision electroweak measurements. Such bounds will thus only be found after more detailed phenomenology of the model is worked out, but we should expect at least $k \gtrsim 10$TeV.

Note that we have no control over the scale $\Lambda_{SU(5)}$; it depends only on the particle content of the model and the gauge coupling constant at some reference energy. The coupling constant will be fixed by requiring consistency with the low-energy coupling constants, and thus $\Lambda_{SU(5)}$ is fixed. Should it turn out to be too low to satisfy the conditions discussed above, that would disprove the model in its current form. Unfortunately we don't yet know how to calculate the confinement scale of a 5D gauge theory, if indeed it is confining.

One quantity which we can easily calculate is the fundamental scale of gravity, M, in terms of the 4D Planck mass, as per section 5.2. We have simply

$$\begin{aligned} M_{Pl}^2 = 2M^3 \int dy\, e^{-2\sigma(y)} &= 2M^3 \int dy\, \cosh^{-a}(ky) \\ &= \frac{2M^3\sqrt{\pi}\Gamma(\frac{a}{2})}{k\Gamma(\frac{1+a}{2})} \end{aligned} \tag{8.19}$$

Because we know the value of M_{Pl}, not M, it is more convenient to express this as

$$M^3 = \frac{\Gamma(\frac{1+a}{2})}{2\sqrt{\pi}\Gamma(\frac{a}{2})} k M_{Pl}^2 \tag{8.20}$$

We can see from this expression that the hierarchy problem is not dealt with in this model. To understand this, note that a hierarchy problem arises if any scalar masses are much smaller than M. Therefore the mass scale k should be $\sim M$, in which case the above equation becomes

$$M^2 \sim \frac{\Gamma(\frac{1+a}{2})}{2\sqrt{\pi}\Gamma(\frac{a}{2})} M_{Pl}^2 \tag{8.21}$$

Therefore if we want M to be much smaller than M_{Pl}, we require $a \ll 1$ ($\Gamma(a) \to \infty$ as $a \to 0$). But a is the ratio of a scalar field parameter ($v^{2/3}$) to M, and therefore must be roughly of order 1 to avoid a different hierarchy. Therefore we cannot arrange for M to be naturally close to the electroweak scale.

The other way in which the hierarchy problem can be addressed is via a similar mechanism to the RS1 model, where Planck scale fundamental parameters get 'warped down' to the electroweak scale. It is in fact possible that this could happen here, because the VEV of the Standard Model Higgs will be concentrated away from $y = 0$ (at the minimum of the potential in figure 8.2), conceivably acting as the 'TeV' brane and yielding a model similar to that in Ref. [47]. Unfortunately, this doesn't work out – arranging for the minimum of the potential to be far enough away from $y = 0$ requires fine tuning.

8.6 Coupling constant unification

As is the case with any grand unified theory, ours is only consistent if the Standard Model gauge coupling constants unify at some high scale. As discussed in section 4.2.2, the coupling constants do not unify if only the Standard Model particles are present. Our theory contains extra particles though, in the form of the Kaluza-Klein excitations of the various fields, and therefore we might hope to achieve unification. The masses of these modes[2] will depend on the choice of parameters, which we have not considered in detail yet. We can make some preliminary comments though.

Firstly, if extra particles are added which form a complete multiplet of $SU(5)$, they have no affect on the question of unification. The reason for this is simple – if adding extra $SU(5)$ multiplets contributed differently to the running of the coupling constants of different subgroups, this would violate the gauge symmetry. In our model however, the $SU(5)$ symmetry is broken by the background such that different members of the *same* multiplet will generically have Kaluza-Klein excitations of *different* mass. Therefore at certain energy scales, only part of the multiplet will contribute to the beta functions, and the above argument doesn't apply. It is this which gives us hope of improving the unification behaviour. What we hope for is that the Kaluza-Klein modes of those fields which tend to improve unification are lighter, and therefore contribute over a wider range of energies, than those that do the opposite.

We can easily calculate the individual contributions to the gauge beta functions from the KK modes of the fermions. First we note that although the zero modes are of specific chirality, all the KK excitations are Dirac particles, and thus contribute twice as much to the beta functions. If we denote by δb_i the

[2]We expect that the resonant modes will make essentially the same contribution to coupling constant running as discrete KK excitations would.

contribution of a single mode to the beta function co-efficient for g_i, defined in equation (4.21), then the various Standard Model multiplets in $\Psi_{5,10}$ give[3]

$$\begin{array}{rlll} \underline{L}: & \delta b_1 = \frac{1}{5} & \delta b_2 = \frac{1}{3} & \delta b_3 = 0 \\ \underline{D}': & \delta b_1 = \frac{2}{15} & \delta b_2 = 0 & \delta b_3 = \frac{1}{3} \\ \underline{U}': & \delta b_1 = \frac{8}{15} & \delta b_2 = 0 & \delta b_3 = \frac{1}{3} \\ \underline{Q}: & \delta b_1 = \frac{1}{15} & \delta b_2 = 1 & \delta b_3 = \frac{2}{3} \\ \underline{E}': & \delta b_1 = \frac{2}{5} & \delta b_2 = 0 & \delta b_3 = 0 \end{array} \tag{8.22}$$

All matter multiplets make a positive contribution to all the beta function co-efficients i.e. cause all graphs in figure 4.3 to 'tilt downwards' more. Therefore the multiplets which improve unification are those which contribute more to b_2 than the other beta function co-efficients. Naïvely then, we want the Kaluza-Klein excitations of L to be lighter than those of D', and the excitations of Q to be lighter than those of U' and E'. This will tend to improve unification, while any other scenario will tend to worsen it.

Of course we also have the scalar fields and gauge fields in our theory. We don't expect χ to contribute to the gauge beta functions, because its components which are not eaten via the Higgs mechanism gain large masses due to its large VEV. Components of the field Φ will contribute as follows:

$$\begin{array}{rlll} \underline{\Phi}_w: & \delta b_1 = \frac{1}{20} & \delta b_2 = \frac{1}{12} & \delta b_3 = 0 \\ \underline{\Phi}_c: & \delta b_1 = \frac{1}{30} & \delta b_2 = 0 & \delta b_3 = \frac{1}{12} \end{array} \tag{8.23}$$

Therefore we want excitations of Φ_w to be lighter than those of Φ_c to improve the prospects of gauge unification. We note however that scalar excitations contribute only one quarter of what fermion excitations in the same representation do, so are less important to the question of unification.

We have not yet considered Kaluza-Klein excitations of the gauge fields, as it is unclear how to analyse these in the presence of the Dvali-Shifman mechanism.

[3]Incidentally, it is easy to check from these numbers that a complete $SU(5)$ multiplet does indeed contribute the same amount to each of the three beta functions.

8.7 Open questions

The theory we have presented is still in the embryonic stage; there are a number of places it could go wrong, and no detailed phenomenology has yet been done. In particular, there are many issues to do with appealing to the Dvali-Shifman mechanism for gauge boson localisation. Here we list some of the obvious next steps to take in the development of this model:

- Study whether or not non-Abelian gauge theories really are confining in 5D. Work out at least qualitatively how to calculate the 5D confinement scale $\Lambda_{SU(5)}$.

- Find (probably numerically) a stable solution of the coupled Einstein and Klein-Gordon equations which includes the VEV of the electroweak Higgs. Identify the Standard Model fermions as light resonances in the fermion spectra, and calculate their lifetime on the brane.

- Fit parameters to reproduce the Standard Model quantitatively at low energies, taking into account 'running' from the unification scale down to the electroweak scale.

- Calculate the low-energy scalar spectrum. In particular, the kink-like field η and the 4D scalar part of gravity mix, and may yield light scalar modes which mediate phenomenologically unacceptable Yukawa interactions.

- Analyse the stability of the domain wall solutions with gravity, taking into account mixing between the scalar fields and 4D scalar gravitational degrees of freedom.

- Calculate the masses of the broken $SU(5)$ gauge bosons (is it sufficient to do this at tree-level?). Find a way to calculate the effects of Kaluza-Klein excitations of the Standard Model gauge bosons.

Chapter 9

Conclusion

Our goal with this project was to develop a realistic field-theoretical model of particle physics in $4+1$ spacetime dimensions. We decided it was most natural if all dimensions were treated on the same footing initially, so chose all dimensions to be non-compact. We then performed a fairly extensive study of these types of models, to confirm that $3+1$ dimensional phenomenology might be reproduced. This allowed us to identify the most promising mechanisms for localising the various fields. The only known way of obtaining Newtonian $1/r^2$ gravity in this context is the Randall-Sundrum idea of warped extra dimensions. Furthermore the only way we have found to (potentially) localise non-Abelian gauge fields and retain gauge universality is that suggested by Dvali and Shifman in which a non-Abelian gauge symmetry holds in the bulk but is broken on the brane.

To implement both the Randall-Sundrum and Dvali-Shifman ideas, we introduced an $SU(5)$ gauge symmetry, and considered an adjoint scalar and a singlet scalar coupled to gravity. We found a domain wall solution to this system which reproduces RS-like gravity and breaks $SU(5)$ to the Standard Model gauge group inside the wall. We then introduced the minimal set of fermion and scalar fields necessary to contain the Standard Model spectrum, and performed some preliminary analysis of this model.

Our theory has the same field content as the original Georgi-Glashow $SU(5)$ model, plus the real gauge-singlet scalar field η. It has however the potential to alleviate a number of undesirable features of the original model:

- Unwanted fermion mass relations are not obtained. The fermions from any multiplet are localised at different positions in the extra dimension, depending on their hypercharge. Because their bare Yukawa couplings to the Standard Model Higgs come from overlap integrals along the extra dimension, they will therefore be generically different.

- Proton decay due to the coloured scalar triplet is generically suppressed, thanks to the splitting of the fermion fields along the extra dimension. This is a possible solution to the doublet-triplet splitting problem; the triplet is relatively light, but its interactions are exponentially small.

- The existence of Kaluza-Klein excitations of the various fields changes the running of the gauge coupling constants, so they might truly unify at high energies.

We cannot draw any firm conclusions about the above issues until after parameter fitting and detailed phenomenological study. However these are all promising aspects of the model, and follow naturally from the minimal assumptions we made in its development. Certainly there is justification for pursuing the idea further, and working out in greater detail the consequences of this theory.

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Appendix A

Potential analysis

The potential for our $SU(5)$ model presented in part III is

$$V(\eta,\chi) = (c\eta^2 - \mu^2)\mathrm{Tr}(\chi^2) + \lambda_1[\mathrm{Tr}(\chi^2)]^2 + \lambda_2\mathrm{Tr}(\chi^4) + l(\eta^2 - v^2)^2. \quad \text{(A.1)}$$

The boundary conditions for our solution are $(\eta,\chi) \to (\pm v, 0)$ as $y \to \pm\infty$. By inspection, $V(\pm v, 0) = 0$, so we want our potential to be positive for all other values of (η,χ), thus ensuring that our solution will be topologically stable. We will now find conditions on the parameters that guarantee this to be the case. First of all, we take all parameters to be positive; this automatically guarantees that the potential is bounded from below.

The argument proceeds as follows. In (A.1), only the term proportional to $\mathrm{Tr}(\chi^2)$ can be negative, which occurs if and only if $c\eta^2 - \mu^2 < 0$. Our task then is to find the minimum of V on the interval $0 \leq \eta^2 < \frac{\mu^2}{c}$, and ensure it is positive. We can obtain one necessary condition immediately. Notice that if $\eta = v$, the last term in V is zero, so V will be negative for small non-zero values of χ unless we demand $cv^2 - \mu^2 \geq 0$. We assume this is true from now on.

For any given value of η^2, we can minimise V by treating it as a function of χ only. Fortunately, this problem has already been solved by Li [48]. For a fixed value of η^2, the minimum of the potential is,

$$\tilde{V} = \frac{-(c\eta^2 - \mu^2)^2}{4\tilde{\lambda}} + l(\eta^2 - v^2)^2 \quad \text{(A.2)}$$

where we have defined $\tilde{\lambda} \equiv \lambda_1 + \frac{7}{30}\lambda_2$.

Now we minimise $\tilde{V}$ on the given interval. First check the end-points. For $\eta^2 = 0$, we get,

$$\tilde{V} = \frac{-\mu^4}{4\tilde{\lambda}} + lv^4 \tag{A.3}$$

So we require,

$$4l\tilde{\lambda}v^4 - \mu^4 > 0 \tag{A.4}$$

For $\eta^2 = \frac{\mu^2}{c}$, we get,

$$\tilde{V} = l\left(\frac{\mu^2}{c} - v^2\right)^2 \tag{A.5}$$

This is manifestly non-negative, and will be positive if we strengthen our earlier condition to a strict inequality: $cv^2 - \mu^2 > 0$.

Now we must check for critical points in the interior of the interval. Differentiating $\tilde{V}$ with respect to η^2 tells us there is a critical point at,

$$\eta^2 = \frac{4l\tilde{\lambda}v^2 - c\mu^2}{4l\tilde{\lambda} - c^2} \tag{A.6}$$

and the value of the potential at this point is,

$$\tilde{V} = l\frac{(cv^2 - \mu^2)^2}{c^2 - 4l\tilde{\lambda}} \tag{A.7}$$

It is not immediately obvious, but this does *not* give any non-trivial constraints on our parameters. To see this, notice that for the value of η^2 at the critical point, we have,

$$c\eta^2 - \mu^2 = -\frac{4l\tilde{\lambda}(cv^2 - \mu^2)}{c^2 - 4l\tilde{\lambda}} \tag{A.8}$$

Therefore it lies in the relevant interval only if $c^2 - 4l\tilde{\lambda} > 0$. If this is the case, then by inspection $V > 0$ at the critical point. Therefore there are no additional constraints to impose on our parameters.

In summary, the potential has the form we want if all parameters are positive, and satisfy,

$$cv^2 - \mu^2 > 0 \qquad 4l\tilde{\lambda}v^4 - \mu^4 > 0 \tag{A.9}$$

A.1 The consistency of our solution

In Chapter 7 we presented an analytical solution to the coupled Einstein and Klein-Gordon equations which relied on certain relations being satisfied by the parameters of the potential. For this to be sensible, we need to check that these relations don't contradict the inequalities given above.

For our analytic solution to hold the following need to be true:

$$
\begin{aligned}
c &= 2l + \frac{\tilde{\lambda}}{2} && \text{(A.10)}\\
\mu_\chi^2 &= \frac{lv^2}{1+a} + \frac{\tilde{\lambda}}{4}\left(\frac{3+4a}{1+a}\right). && \text{(A.11)}
\end{aligned}
$$

This gives

$$
cv^2 - \mu_\chi^2 = v^2(4l - \tilde{\lambda})\left(\frac{1+2a}{1+a}\right) \tag{A.12}
$$

Therefore the inequality simply becomes $4l - \tilde{\lambda} > 0$. For the second expression we get a much more complicated expression:

$$
\begin{aligned}
&4l\tilde{\lambda}v^4 - \mu_\chi^4 = \\
&\frac{v^4}{(1+a)^2}\left\{\tilde{\lambda}^2\left[\left(\frac{5}{4} + 3a - 2a^2\right)^2 + \frac{1}{16}(3+4a)^2\right] - \left[l - \left(\frac{5}{4} + 3a - 2a^2\right)\tilde{\lambda}\right]^2\right\}
\end{aligned}
\tag{A.13}
$$

This is a particularly convenient way to write this quantity, because the first term is manifestly positive for all positive values of a. The second term can be made arbitrarily small by imposing

$$
l \simeq (\frac{5}{4} + 3a - 2a^2)\tilde{\lambda}. \tag{A.14}
$$

As long as $a \leq 7/4$, this won't contradict $4l - \tilde{\lambda} > 0$. Having imposed this approximate equality, the complicated expression above is positive, so our solution doesn't contradict the necessary inequalities.

Note that the above approximate equality is a very loose requirement – there will in fact be a reasonably large region of parameter space in which the inequalities (A.8) hold, even given the relations needed for our analytic solution.

Appendix B

Fermions in curved spacetime

In theories with warped extra dimensions we are working with a curved background spacetime, and thus need to know how to take derivatives of various fields in curved spacetime. For vector fields and higher rank tensors this is easy – we simply use the Levi-Civita connection. Spinors however are not tensors, so we need to develop a different formalism to deal with them. The following has the additional benefit of being a very clear and physically intuitive alternative way of presenting the basic differential geometry needed for general relativity.

At any point x in a general n-dimensional spacetime, coordinates can be chosen such that $g_{MN}(x) = \eta_{MN}$. This implies that there exists a set of n vectors $\{V_0{}^M(x), \ldots, V_n{}^M(x)\}$ satisfying

$$g_{MN}(x) V_A{}^M(x) V_B{}^N(x) = \eta_{AB}. \tag{B.1}$$

This is essentially just an orthonormal basis for the tangent space at x. If we choose this basis to vary smoothly in some coordinate patch, we get a set of n local vector *fields* satisfying[1] $g_{MN} V_A{}^M V_B{}^N = \eta_{AB}$ The fields $\{V_A{}^M\}$ are

[1]Generally speaking this cannot be done throughout the spacetime. If such a set of vector fields exists globally it implies triviality of the tangent bundle – a non-trivial topological condition.

collectively referred to as the 'vielbein' (or vierbein or tetrad in 4D), meaning "many legs" in German.

The vielbein is certainly not unique. Noting that Eq. (B.1) defines the vielbein, we can take any smoothly spacetime-dependent choice of Lorentz transformation, $\Lambda_A{}^B(x)$, and obtain an equally good choice,

$$\tilde{V}_A{}^M(x) = \Lambda_A{}^B(x) V_B{}^M. \tag{B.2}$$

This redundancy in choice of the vielbein is precisely the local Lorentz invariance of general relativity. The vielbein can be thought of as carrying one 'generally covariant' index and one 'internal Lorentz' index. If by definition we raise and lower the Lorentz indices with η_{AB}, η^{AB}, we can easily prove the following identities:

$$\begin{aligned} V^A{}_M V_B{}^M = \delta^A_B &\implies V_{AM} V_B{}^M = \eta_{AB} \\ V^A{}_M V_A{}^N = \delta^N_M &\implies V^A{}_M V_{AN} = g_{MN} \end{aligned} \tag{B.3}$$

Usually components of tensors are expressed in a coordinate basis, but we can instead give the components of any tensor in terms of the vielbein. For a vector field T, we write

$$T^A = V^A{}_M T^M. \tag{B.4}$$

The generalisation to tensors of any rank then follows. Notice that these components are invariant under general coordinate transformations, but under local Lorentz transformations they transform as in the flat space case.

So how are we to deal with covariant differentiation in this formalism? When working with components in the coordinate basis, the Christoffel symbols compensate for the way the basis vectors vary through spacetime. Here we have the same concept; the vielbein rotates as we move around in spacetime, and this information is contained in its covariant derivatives. Because the vielbein gives a basis at each point, there exist co-efficients $\omega^A{}_{BM}$ (called connection co-efficients) such that

$$D_M V_B{}^N = \omega^A{}_{BM} V_A{}^N \tag{B.5}$$

From this follows the formula used in practice to calculate the connection co-efficients:

$$\omega^A{}_{BM} = V^A{}_N D_M V_B{}^N. \tag{B.6}$$

If we lower the first index with η, then it is easy to verify that the connection co-efficients satisfy $\omega_{ABM} = -\omega_{BAM}$. This is no coincidence, for the generators of the Lorentz group in n spacetime dimensions are $n \times n$ matrices $a^A{}_B$ satisfying $a_{AB} = -a_{BA}$ where $a_{AB} \equiv \eta_{AC} a^C{}_B$; so each matrix ω_M belongs to the Lie algebra of the Lorentz group. It can further be shown that under a spacetime dependent local Lorentz transformation Λ, ω_M transforms according to

$$\omega_M \to \Lambda \omega_M \Lambda^{-1} + (\partial_M \Lambda)\Lambda^{-1}. \tag{B.7}$$

It is the connection co-efficients which tell us how to differentiate a tensor field given in terms of its 'flat' components. If we take the covariant derivative of Eq. (B.4) we find

$$D_M T^A = \partial_M T^A + \omega^A{}_{BM} T^B. \tag{B.8}$$

The preceding paragraph contains some very familiar equations – ω_M is playing the role of a gauge field, where the gauge group is the Lorentz group $SO(n-1,1)$. This is not surprising, as we are dealing with a local symmetry described by a Lie group, just as in gauge theory. We can thus use our knowledge of gauge theory to construct the covariant derivative of a field of arbitrary spin, by substituting the relevant representation of the Lorentz group.

Under an infinitesimal Lorentz transformation $\Lambda^A{}_B = \delta^A_B + \epsilon^A{}_B$ a spinor transforms like

$$\psi \to \left(1 + \frac{1}{4}\epsilon^A{}_B \eta_{AC} \gamma^C \gamma^B\right)\psi. \tag{B.9}$$

Therefore the covariant derivative appropriate for a spin-1/2 field is simply

$$D_M \psi = \left(\partial_M + \frac{1}{4}\omega^A{}_{BM} \eta_{AC} \gamma^C \gamma^B\right)\psi, \tag{B.10}$$

which allows us to construct the Dirac Lagrangian:

$$\mathcal{L}_D = i\overline{\psi}\Gamma^M \left(\partial_M + \frac{1}{4}\omega^A{}_{BM} \eta_{AC} \gamma^C \gamma^B\right)\psi, \tag{B.11}$$

where Γ^M are the curved space gamma matrices, which generate the Clifford algebra associated with the metric g_{MN}: $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$. They are related to the flat space gamma matrices via the vielbein i.e. $\Gamma^M = \Gamma^A V_A{}^M$.

Appendix C

One-loop calculations

At this stage, the only reason we are interested in quantum corrections is the 'running' of the gauge coupling constants that they cause. The calculation of the one-loop beta function for a gauge theory with fermions is performed explicitly in Ref. [34]. We will therefore not reproduce the entire calculation, but elaborate on some of the more interesting points, as well as calculating the contribution due to adding scalars to the theory.

The formula for the one-loop beta function is equation (4.18), so we need to find the counterterms to leading order. Instead of doing the whole calculation, we will take the contributions of most diagrams from Ref. [34]. Notice that the contributions from various diagrams simply add together to give the final counterterms, so we will use the notation $\Delta\delta$ for a partial contribution to a counterterm, then add them all at the end.

We take from Ref. [34] the values for all the diagrams in figure C.1. These are calculated in Feynman gauge $\xi = 1$, and give

$$\Delta\delta_3 = C_2(\mathrm{Ad})\frac{5g^2}{48\pi^2}\log M^2 + \text{M-independent terms} \tag{C.1}$$

$$\Delta\delta_2 = -C_2(\mathbf{r})\frac{g^2}{16\pi^2}\log M^2 + \text{M-independent terms} \tag{C.2}$$

$$\Delta\delta_1 = -[C_2(\mathbf{r}) + C_2(\mathrm{Ad})]\frac{g^2}{16\pi^2}\log M^2 + \text{M-independent terms} \tag{C.3}$$

where Ad is the adjoint representation, which the gauge bosons belong to, and $\mathbf{r}$ the representation to which the fermion in question belongs.

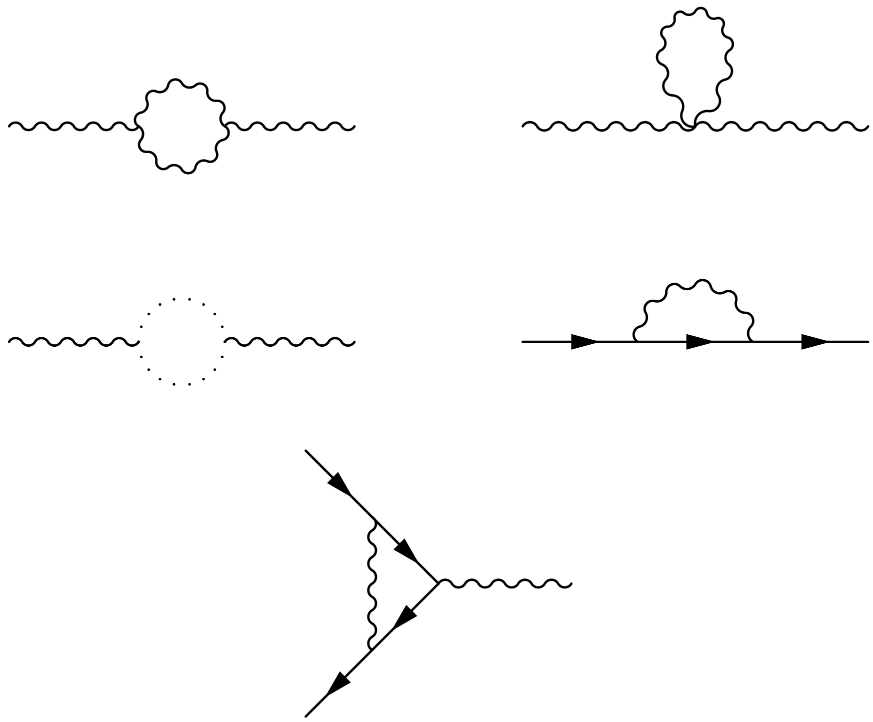

Figure C.1: Some one-loop corrections in non-Abelian gauge theory.

C.1 Scalar contribution to vacuum polarisation

If a charged scalar is added to a gauge theory, then obviously as well as fermions loops, scalar loops will contribute to the vacuum polarisation (gauge boson self-energy, or propagator corrections). We will perform the calculation of this contribution and include many of the steps in the hope that this appendix will serve as a good reference for performing one-loop Feynman diagram calculations.

The Lagrangian for a scalar field ϕ in representation $\mathbf{r}$ of a gauge group with generators $\{t^a\}$ is

$$\begin{aligned}\mathcal{L} =&(\partial_\mu \phi^\dagger + igA^a_\mu \phi^\dagger t^a_{\mathbf{r}})(\partial^\mu \phi - igA^{b\mu} t^b_{\mathbf{r}} \phi)\\ =&\partial_\mu \phi^\dagger \partial^\mu \phi + igA^a_\mu(\phi^\dagger t^a_{\mathbf{r}} \partial^\mu \phi - \partial^\mu \phi^\dagger t^a_{\mathbf{r}} \phi) + g^2 A^a_\mu A^{b\mu} \phi^\dagger t^a_{\mathbf{r}} t^b_{\mathbf{r}} \phi,\end{aligned} \tag{C.4}$$

from which we derive the interaction vertices shown in figure C.2. The two interaction vertices give rise to two diagrams contributing to the vacuum polarisation at one-loop level, as shown in figure C.3. If either of these diagrams are calculated independently, they diverge quadratically, which would violate

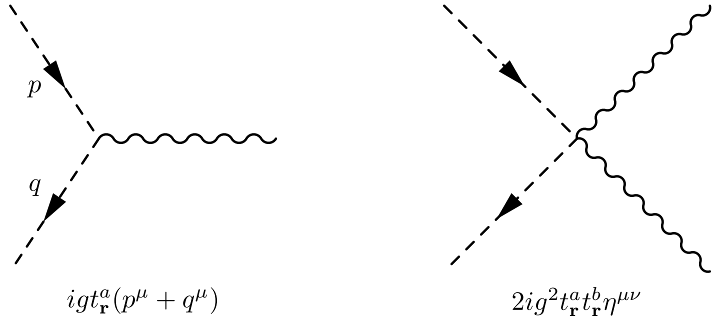

Figure C.2: Feynman rules for coupling between scalar and gauge fields. All momenta flow in the direction of the arrows.

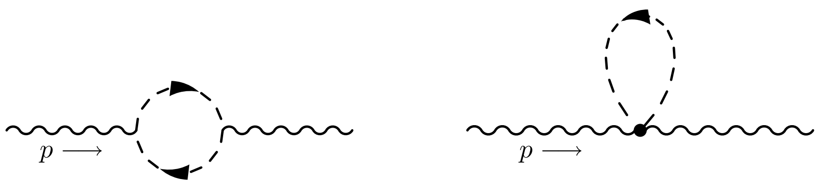

Figure C.3: The two one-loop contributions of a charged scalar to the vacuum polarisation.

gauge invariance by giving mass to the gauge bosons. However, both must of course be considered, and if we add them before we evaluate the loop integrals, we obtain

$$\begin{aligned}
&\frac{g^2}{(2\pi)^4}\mathrm{Tr}[t^a_{\mathbf{r}} t^b_{\mathbf{r}}]\int d^4l \left\{ \frac{(2l^\mu + p^\mu)(2l^\nu + p^\nu)}{(p+l)^2 l^2} - \frac{2\eta^{\mu\nu}}{l^2} \right\} \\
=&\frac{g^2}{(2\pi)^4} C(\mathbf{r})\delta^{ab}\int d^4l \; \frac{(2l^\mu + p^\mu)(2l^\nu + p^\nu) - 2(p+l)^2\eta^{\mu\nu}}{(p+l)^2 l^2}
\end{aligned} \tag{C.5}$$

where we are using $l^2 \equiv l \cdot l$ for a Lorentz four-vector l. We now introduce a Feynman parameter x, using the identity

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[(1-x)A + xB]^2} \tag{C.6}$$

so that our amplitude becomes

$$\frac{g^2}{(2\pi)^4} C(\mathbf{r})\delta^{ab}\int_0^1 dx \int d^4l \; \frac{(2l^\mu + p^\mu)(2l^\nu + p^\nu) - 2(p+l)^2\eta^{\mu\nu}}{[l^2 + 2xp\cdot l + xp^2]^2} \tag{C.7}$$

We now complete the square in the denominator by defining $k = l + xp$. Note that $d^4k = d^4l$ and that because the denominator is even in k, we can throw away any terms in the numerator which are odd in k. We get

$$\frac{g^2}{(2\pi)^4}C(\mathbf{r})\delta^{ab}\int_0^1 dx \int d^4k \, \frac{4k^\mu k^\nu + (1-2x)^2 p^\mu p^\nu - 2k^2\eta^{\mu\nu} - 2(1-x)^2 p^2 \eta^{\mu\nu}}{[k^2 + x(1-x)p^2]^2}. \tag{C.8}$$

We wish to use dimensional regularisation, so at this point we let k live in a d-dimensional spacetime. By symmetry, the term in the integral proportional to $k^\mu k^\nu$ must evaluate to be proportional to $\eta^{\mu\nu}$. Contracting indices we determine that the constant of proportionality is k^2/d, and thus replace $k^\mu k^\nu$ by $\eta^{\mu\nu}k^2/d$ in our integrand. This leaves us with an integral which would be fairly straightforward to evaluate except for the Lorentzian signature. Now is when we need to remember the $+i\epsilon$ term which we have been leaving out of the denominator. If we consider first the integral over the component k^0, this $+i\epsilon$ means that there are no poles in the first and third quadrants of the complex k^0 plane, allowing us to rotate the contour anti-clockwise by $\pi/2$, onto the imaginary axis. This is equivalent to a change of variables $k \to k_E$ with $k_{Ei} = k_i$ and $k_E^0 = ik^0$. Our integrand then looks like a rotationally invariant function in d-dimensional Euclidean space, allowing us to go to polar coordinates and obtain

$$\frac{ig^2}{(2\pi)^4}C(\mathbf{r})\delta^{ab}\int_0^1 dx \int d\Omega_d \int_0^\infty dr \frac{1}{[r^2 - x(1-x)p^2]^2}\Big\{2\left(1-\frac{2}{d}\right)\eta^{\mu\nu}r^{d+1} + [(1-2x)^2 p^\mu p^\nu - 2(1-x)^2 p^2 \eta^{\mu\nu}]r^{d-1}\Big\}, \tag{C.9}$$

where $d\Omega_d$ is the element of d-dimensional solid angle. Defining[1] $\Delta = x(1-x)p^2$ and using the results from appendix D as well as the defining property $z\Gamma(z) = \Gamma(z+1)$ of the gamma function, we get

$$\frac{ig^2}{16\pi^2}C(\mathbf{r})\delta^{ab}\int_0^1 dx \, (\pi\Delta)^{\frac{d}{2}-2}\Gamma(2-\frac{d}{2})[2(1-x)(2x-1)p^2\eta^{\mu\nu} + (2x-1)^2 p^\mu p^\nu]. \tag{C.10}$$

Note that if we define $y = x - 1/2$, we can throw away any terms odd in y. This leaves us with

$$-\frac{ig^2}{4\pi^2}C(\mathbf{r})\delta^{ab}[p^2\eta^{\mu\nu} - p^\mu p^\nu]\int_0^1 dx \, (x-\frac{1}{2})^2(\pi\Delta)^{\frac{d}{2}-2}\Gamma(2-\frac{d}{2}). \tag{C.11}$$

[1] Notice that for integral to make sense, Δ must be positive, and therefore p must be spacelike. Renormalising at timelike momenta brings extra complications, which we don't deal with here.

We can see that we hit no singularities in increasing d from a small positive number to nearly 4, so let $d = 4 - \epsilon$, and use the expansions

$$\begin{aligned} \Gamma(\frac{\epsilon}{2}) &= \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \\ (A)^{-\frac{\epsilon}{2}} &= 1 - \frac{\epsilon}{2}\log A + \mathcal{O}(\epsilon^2), \end{aligned} \tag{C.12}$$

where γ is the Euler-Mascheroni constant. We renormalise at spacelike momentum, so set $p^2 = -M^2$. Then we need only perform some simple integrals over x to obtain our final expression for the amplitude:

$$-\frac{ig^2}{48\pi^2}C(\mathbf{r}_s)\delta^{ab}[p^2\eta^{\mu\nu} - p^\mu p^\nu]\left\{\frac{2}{\epsilon} - \log M^2 - \log\pi - \gamma + \frac{8}{3} + \mathcal{O}(\epsilon)\right\}. \tag{C.13}$$

Therefore the contribution $\Delta\delta_3$ of the scalar to the gauge boson propagator counterterm

$$\Delta\delta_3 = \frac{g^2}{48\pi^2}C(\mathbf{r}_s)\log M^2 + \text{M-independent terms.} \tag{C.14}$$

In section 4.2, we showed that if we calculate the gauge beta function from the fermion-gauge boson vertex it depends on the counterterms for the vertex and fermion propagator as well as the gauge boson propagator. It is conceivable that interactions with the scalar could also contribute to these counterterms, through a Yukawa coupling to the fermion, thus further affecting the beta function via the loop diagrams in figure C.4. In fact we can argue that any such contributions must cancel. If there are multiple fermions in the theory, their gauge coupling constants must all be equal by gauge invariance, however there is no reason they could not have vastly different Yukawa couplings to ϕ. Therefore if ϕ made a net contribution to the beta function through its Yukawa couplings, we would get a different beta function for each fermion gauge coupling, and this would violate gauge invariance[2]. Our intuition is confirmed by an explicit calculation (not presented here) for the case when the scalar is uncharged under the gauge group.

[2]Thanks go to A. Kobakhidze for this observation.

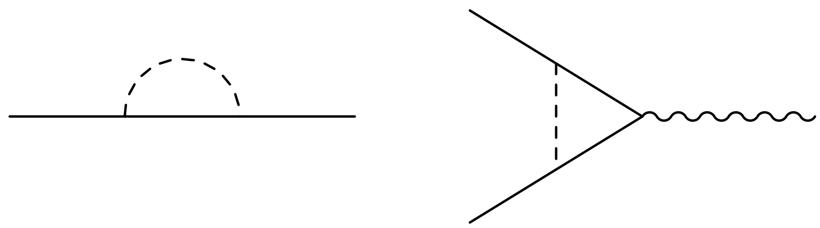

Figure C.4: One-loop corrections to the fermion propagator and gauge-fermion vertex, due to a Yukawa coupling term.

C.2 Vacuum polarisation by Weyl fermions

In section 4.2 we calculated vacuum polarisation by a Dirac fermion in a $U(1)$ gauge theory. For a more complicated gauge group we simply pick up the group theory factor $C(\mathbf{r})$ from tracing over the fermion loop, but there is another complication – we know that the left- and right-handed Weyl components of a Dirac fermion can be in different representations, and indeed this is true in the Standard Model! Therefore we need to work out the vacuum polarisation from a Weyl fermion. We will verify the intuitive result that it contributes half what a Dirac fermion would, corresponding to it having half the degrees of freedom.

Suppose a gauge field couples to just the (say) left-handed part of a fermion. Then the gauge interaction term is $gA^a_\mu\overline{\psi}\gamma^\mu\frac{1}{2}(1-\gamma^5)t^a_{\mathbf{r}}\psi$, and the amplitude for the vacuum polarisation diagram (looking like figure 4.2) becomes

$$\begin{aligned}&(ig)^2C(\mathbf{r})\delta^{ab}\int\frac{d^4l}{(2\pi)^4}(-1)\mathrm{Tr}\left[\frac{i\not{l}}{l^2}\gamma^\mu\frac{1}{2}(1-\gamma^5)\frac{i(\not{l}-\not{p})}{(l-p)^2}\gamma^\nu\frac{1}{2}(1-\gamma^5)\right]\\ =&\frac{g^2}{(2\pi)^4}C(\mathbf{r})\delta^{ab}\int d^4l\frac{1}{2}\mathrm{Tr}\left[\frac{\not{l}}{l^2}\gamma^\mu\frac{(\not{l}-\not{p})}{(l-p)^2}\gamma^\nu(1-\gamma^5)\right],\end{aligned}\tag{C.15}$$

where we have anti-commuted γ^5 past two gamma matrices, and used the simple identity $(1-\gamma^5)^2=2(1-\gamma^5)$. Comparing this to equation (4.11), we can see that it is half the contribution from a Dirac fermion, plus an extra term involving γ^5. We can demonstrate that this extra term vanishes as follows: The denominator of the integrand is even under $l\to p-l$, therefore we need only keep terms in the numerator which are also even under this replacement. The term containing

γ^5 can therefore be manipulated as follows:

$$\begin{aligned} \mathrm{Tr}[\not{l}\gamma^\mu(\not{l}-\not{p})\gamma^\nu\gamma^5] &= l_\tau(l_\kappa - p_\kappa)\mathrm{Tr}[\gamma^\tau\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^5] \\ &\to \frac{1}{2}\{l_\tau(l_\kappa - p_\kappa) + l_\kappa(l_\tau - p_\tau)\}[-4i\epsilon^{\tau\mu\kappa\nu}] \\ &= 0, \end{aligned} \tag{C.16}$$

where in the second line we have replaced $l_\tau(l_\kappa - p_\kappa)$ with its part even in $l \to p - l$, and the last line vanishes because we are contracting a quantity even in τ, κ with one odd in the same indices.

So we have shown the pleasing result that a Weyl fermion contributes to the vacuum polarisation exactly half what a Dirac fermion does. Therefore we can take equation (4.14) and conclude that the change in δ_3 due to a Weyl fermion in representation $\mathbf{r}$ is given by

$$\Delta\delta_3 = C(\mathbf{r})\frac{e^2}{24\pi^2}\log M^2 + \text{M-independent terms} \tag{C.17}$$

C.3 The beta function of a non-Abelian gauge theory

Adding up the various contributions to the counterterms, and applying equation (4.18) which gives the one-loop beta function in terms of derivatives of counterterms, we obtain

$$\beta(g) = -\frac{g^3}{48\pi^2}[11C_2(\mathrm{Ad}) - \sum_f 2C(\mathbf{r}_f) - \sum_s C(\mathbf{r}_s)], \tag{C.18}$$

where f labels the *Weyl* fermions in the theory, $\mathbf{r}_f$ their representations, and s labels the *complex* scalars, with $\mathbf{r}_s$ their representations. The standard expression for an $SU(N)$ gauge theory comes from noting that the quantity $C_2(\mathrm{Ad})$ is simply N for $SU(N)$.

Appendix D

Dimensional Regularisation

To perform quantum field theory calculations and obtain finite answers, it is necessary to first 'regularise' the theory, and then renormalise quantities such that when the regulator is removed, only a finite result remains. Many 'obvious' methods of regularisation, such as introducing a (Euclidean) momentum cutoff, violate symmetries of the theory such as Lorentz invariance or, more importantly, gauge invariance. One scheme which preserves all such symmetries is dimensional regularisation.

To apply dimensional regularisation, we replace the spacetime dimension in loop integrals by an arbitrary number d, and take the limit $d \to 4$ only in the last step. This works because the volume form of d-dimensional space is proportional to r^{d-1}, where r is a radial coordinate, so integrals are less ultraviolet divergent in lower dimensions.

Example

A very typical example of a divergent loop integral is one which given by

$$I = C \int d^4k \frac{1}{(k^2 + \Delta)^2}, \tag{D.1}$$

where C is some constant and Δ is a function of external momenta only. By inspection we can see that this is logarithmically divergent. Replace the spacetime dimension by the arbitrary positive number d, and go to spherical coordinates to obtain

$$I = C \int d\Omega_d \int_0^\infty dr \frac{r^{d-1}}{(r^2+\Delta)^2}, \tag{D.2}$$

where $d\Omega_d$ is the d-dimensional solid angle element. The angular integration simply gives the volume of the $d-1$-sphere, which is

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}. \tag{D.3}$$

The integration over r is achieved by means of a coordinate change:

$$y = \frac{\Delta}{r^2+\Delta}. \tag{D.4}$$

In terms of y,

$$I = C\pi^{\frac{d}{2}} \int_0^1 dy\, y^{\frac{1}{2}(2-d)}(1-y)^{\frac{1}{2}(d-2)}\Delta^{\frac{1}{2}(d-4)}. \tag{D.5}$$

We can now appeal to the result

$$\int_0^1 dz\, z^{\alpha-1}(1-z)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{D.6}$$

which holds for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, to get

$$I = C\pi^{\frac{d}{2}}\Delta^{\frac{1}{2}(d-4)}\Gamma\left(\frac{1}{2}(4-d)\right) \tag{D.7}$$

for $d < 4$. We can see that the divergence of the original integral shows up as the simple pole of the gamma function as its argument approaches zero. This pole can be subtracted by the relevant counterterm to renormalise the theory.

The same technique works for any power of r in the numerator of the initial integrand, and results for larger powers of $(r^2+\Delta)$ in the denominator can be obtained by differentiating with respect to Δ.