Localisation of fields to domain walls

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Abstract

In models with one extra spatial dimension, a brane can be realised as a scalar field domain wall. In order to identify this with our universe, standard model fields need to be localised on the wall by dynamical mechanisms, to explain why we don't observe the extra dimension.

It is well known that fermions can be localised by a Yukawa coupling to the field forming the domain wall. There is as yet no widely accepted technique for localising gauge bosons, but this project explores two possibilities. In the first, gauge bosons are localised by coupling to a second scalar field called the dilaton. It is shown however that the inclusion of the dilaton in the model destroys the localisation of the fermions.

The second model has an $SU(5)$ gauge theory in the bulk, which is broken to the standard model gauge group on the domain wall. It is argued, following Dvali and Shifman [1], that the confinement property of the bulk theory will result in massless standard model fields being trapped on the wall. Stable classical solutions are found for a Higgs field in the adjoint of $SU(5)$ which performs the required symmetry breaking.

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Statement of Contributions

Sections 2-5 are an original review of standard material. Sections 6-8 represent original work done by the author in the course of this project. This was assisted by discussions with Ray Volkas and Damien George. The C code used for the numerical work was originally written by Damien George, but modified and added to by the author.

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Contents

1 Introduction

In recent years there has been a lot of interest in models of particle physics which utilise extra spacetime dimensions. It is possible in many cases to show that physics would nonetheless give the appearance of $3+1$ dimensions at the energy scales thus far probed by experiment.

The original method for 'hiding' the extra dimensions was to postulate that they were curled up to a very small size. In this case, physics acts $(3+1)$-dimensional all the way up to the energy scale corresponding to the inverse of the diameter of these extra dimensions. It seems unnatural, however, for our universe to have three very large spatial dimensions, while the rest are microscopic. We would rather start with a theory in which all dimensions are equivalent, and find *solutions* which result in only three large dimensions.

The basis for this project is a scalar field in 4+1 dimensions which forms a topological defect called a domain wall, which then serves as our $(3+1)$-dimensional universe. The aim is to localise standard model fields to the wall to yield an effective $(3+1)$-dimensional theory. The most difficult problem proves to be localising the gauge bosons. The first approach taken is shown to cause insurmountable problems with the other particles. However, success is achieved with an $SU(5)$ grand unified theory in $4+1$ dimensions, which results in just the standard model fields being trapped on the wall.

In sections 2-4 of this thesis we review the basic properties of the standard model, plus $SU(5)$ theory, and those aspects of extra-dimensional models most relevant to the project. Section 5 is an introduction to a somewhat non-standard approach to general relativity, which is required in what follows. Section 6 consists of calculations and discussion of the dilaton model, which was a failed attempt to localise gauge bosons and fermions simulataneously, while sections 7-8 detail the successful $SU(5)$ model developed, plus some initial discussion about including gravity in the picture.

Appendix A contains a review of the mathematics required to understand gauge theory. Appendix B is a brief explanation of the concept of a topological defect, of which a domain wall is an example, while Appendix C explains the main numerical procedures utilised in the project.

1.1 Notation/conventions

With the exception of section 6, we use the $(+,-,\ldots,-)$ metric signature. The reason for the change in section 6 was to be consistent with [2], from which the work followed directly. The main differences are a change of sign for the kinetic term of a scalar field, and the different definition of the Dirac matrices, which replaces the 'i' in the fermion kinetic term with a -1.

Throughout, upper case roman letters (M, N, ...) denote five-dimensional indices, and lower case greek letters (μ, ν, ...) denote four-dimensional indices. y will always be used for the extra-dimensional coordinate. In sections 5 and 6, upper case roman letters from the *start* of the alphabet (A, B, ...) denote five-dimensional 'flat space indices', and lower case greek letters from the start of the alphabet (α, β, ...) denote four-dimensional flat space indices.

Group generators will always be normalised so that $Tr(\tau^a\tau^b) = \frac{1}{2}\delta_{ab}$. This means for example that the correct normalisation for the kinetic term of a field in the adjoint rep. is $Tr(D_M\chi D^M\chi)$.

The summation convention is used throughout, so unless otherwise noted, repeated indices are always summed over.

2 Gauge Theory and the Standard Model

It is usually said that nature exhibits four different fundamental types of force: gravity, the electromagnetic force, and the strong and weak nuclear forces. Between individual elementary particles[1], the force of gravity is exceedingly weak compared to the others, so for all practical purposes can be ignored. Therefore theoretical particle physics aims to describe the other three forces. The modern description of electromagnetism and the nuclear forces revolves around the concept of symmetry. Everyone is familiar with symmetry in everyday situations – the fact that certain objects are unchanged when rotated or reflected. The symmetries relevant to particle physics are somewhat more abstract, although the basic idea is the same.

The type of symmetry we are interested in was first noticed in the early days of nuclear physics. It was observed that the force which holds the nucleus together seems not to differentiate between protons and neutrons, and that these particles have almost identical mass. Heisenberg proposed a model in which protons and neutrons are states of a single particle called a 'nucleon', which possesses a quantity analogous to the familiar intrinsic spin, called **isospin**. A proton corresponds to isospin $+\frac{1}{2}$, and a neutron to isospin $-\frac{1}{2}$. The nucleon wavefunction is then $N = (p \;\; n)^T$ where p, n are

[1] Leptons and quarks, as far as we know at this time.

the proton and neutron wavefunctions respectively. Mathematically, the proton-neutron symmetry manifests in the fact that the nuclear Hamiltonian can be written in terms of N, and remains unchanged when N is multiplied by any 2×2 special unitary matrix[2]. It is said that the theory 'has an $SU(2)$ symmetry'.

Yang and Mills extended the above idea by 'gauging' the $SU(2)$ symmetry [3], to obtain a dynamical theory of the nuclear force. Although this theory turned out to be incorrect, the idea lies at the very heart of modern descriptions of particle interactions. As such, we now briefly review the ideas of gauge theory, and the standard model of particle physics.

2.1 Gauge theories

(Notation: Greek indices μ, ν etc. are Lorentz indices; Roman indices a, b etc. from the start of the alphabet are Lie algebra indices, and Roman indices i, j etc. from the middle of the alphabet label components of a representation.)

The basic ingredient we start with is a k-tuplet $\Phi = (\phi_1, \dots, \phi_k)$ of fields that is acted on by a k-dimensional representation of some n-dimensional Lie group G. We can take the representation to be irreducible without loss of generality; if it wasn't, we would simply consider an irreducible subrepresentation instead (see appendix A).

Suppose we have a Lagrangian for Φ that is invariant under the action of G. For $U \in G$ we can write $U = exp(-i\theta^a \tau^a)$ where $\{\tau^1, \dots, \tau^n\}$ is a basis for the Lie algebra of G and the θ^a are real numbers. We will adopt the physicists' terminology and refer to the τ's as the group generators. We choose them such that,

$$Tr(\tau^a \tau^b) = \frac{1}{2}\delta_{ab}$$

and define the structure constants C^{abc} by,

$$\left[\tau^a, \tau^b\right] = iC^{abc}\tau^c$$

If the θ^a are constant, it is immediate that when we make the transformation $\Phi \longrightarrow U\Phi = exp(-i\theta^a\tau^a)\Phi$, we also get $\partial_\mu \Phi \longrightarrow U\partial_\mu\Phi$. It is this property that guarantees the invariance of terms such as the kinetic term — $\partial_\mu\Phi\partial^\mu\Phi$ for a scalar field, or $i\overline{\Phi}\gamma^\mu\partial_\mu\Phi$ for a fermion. The **gauge principle** asserts that the Lagrangian should also be invariant under the more general transformation where we allow the group parameters to be function of x, $\theta^a(x)$. In this case, a G-transformation gives,

$$\begin{aligned} \Phi(x) &\longrightarrow U(x)\Phi(x) \\ \partial_\mu\Phi(x) &\longrightarrow U(x)\partial_\mu\Phi(x) + (\partial_\mu U(x))\Phi(x) \end{aligned}$$

It is obvious that the extra term occurring in the second line means that the Lagrangian will *not* be invariant under this transformation. For the gauge principle to be viable, we have to define a new derivative operator D_μ by demanding that if $\Phi(x) \longrightarrow U(x)\Phi(x)$, then $D_\mu\Phi(x) \longrightarrow U(x)D_\mu\Phi(x)$. As long as our Lagrangian was initially invariant under constant G-transformations, it will then be invariant under position-dependent ones.

Intuitively, we could argue that the new derivative operator must have one degree of freedom for each group generator, in order to be able to compensate for the most general possible G-transformation. This is in fact the case. We define the **gauge covariant derivative** by,

$$D_\mu\Phi = (\partial_\mu - igA^a_\mu\tau^a)\Phi$$

where each A^a is a new vector field. The condition $D_\mu(U\Phi) = U(D_\mu\Phi)$ now becomes,

$$(\partial_\mu - igA'^a_\mu\tau^a)U\Phi = U(\partial_\mu - igA^a_\mu\tau^a)\Phi$$

which leads to,

$$A'^a_\mu\tau^a = UA^a_\mu\tau^aU^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Therefore we define **gauge transformations** of the theory to take the form,

$$\begin{aligned} \Phi &\longrightarrow U\Phi \\ A^a_\mu\tau^a &\longrightarrow UA^a_\mu\tau^aU^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \end{aligned}$$

[2]Notice that because the proton and neutron have different electric charges, isospin cannot be an exact symmetry of nature. In the nucleus, however, electromagnetic effects are small, and isospin is a very good *approximate* symmetry.

The covariant derivative we have defined is reminiscent of the minimal coupling prescription for incorporating electromagnetic interactions into quantum mechanics. This is no coincidence. If $G = U(1)$, there is only one group generator, and gauge transformations are just the familiar gauge transformations of Maxwell's theory. Electromagnetism is a gauge theory, with gauge group $U(1)$!

In electromagnetism, A_μ represents the dynamical degrees of freedom of the electromagnetic field itself. To have a complete dynamical theory, we need to introduce a kinetic term for the gauge fields. We will do this in a slightly sophisticated way, but notice that in the case $G = U(1)$, we get the familiar $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, from which Maxwell's equations follow.

Note that although ordinary partial derivatives always commute, gauge covariant derivatives do not. This leads us to define an n-tuplet (remember $n = dim(G)$) of anti-symmetric rank two tensors $F^a_{\mu\nu}$ by,

$$(D_\mu D_\nu - D_\nu D_\mu)\Phi = -igF^a_{\mu\nu}\tau^a\Phi$$

From this we can derive the expression,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu$$

For $U(1)$, $C^{abc} = 0$, so this is the usual definition of the field strength tensor. It is easily proved that under a gauge transformation, $F^a_{\mu\nu}\tau^a \longrightarrow UF^a_{\mu\nu}\tau^a U^{-1}$. The trace of a matrix is unchanged under such a transformation, so we can write down the gauge-invariant gauge-kinetic term,

$$\begin{aligned}\mathcal{L}_{gk} &= -\frac{1}{2}\mathrm{Tr}(F^a_{\mu\nu}\tau^a F^b_{\mu\nu}\tau^b) \\ &= -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\mathrm{Tr}(\tau^a\tau^b) \\ &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\end{aligned}$$

From the definition of F, we see that $\mathcal{L}_{gk}$ contains terms which in quantum perturbation theory yield vertices with three or four gauge bosons, and no other particles. So for non-Abelian gauge groups, the gauge bosons interact with each other, making for much richer dynamics than in the Abelian $U(1)$ case.

We now have the ingredients to construct a fully dynamical theory of interacting particles, where the interaction is mediated by the gauge bosons A^a_μ associated with the group G.

2.2 The standard model of particle physics

The best theoretical picture we currently have of the nuclear forces and electromagnetism is that together they are described by a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$. There is a 'charge' associated with each factor, usually referred to as colour, weak isospin, and weak hypercharge (or sometimes just hypercharge) respectively. Notice that none of these correspond to electric charge; we will see later how electromagnetism fits into this picture. We will not go into the choice of this gauge group here (see eg. [4]), but we do need to know which particles are considered to be elementary, and which representations of the gauge group they belong to.

In the standard model, the basic matter particles are quarks and leptons. The lepton sector contains the electron, its two heavier cousins the muon and the tauon, as well as a neutrino corresponding to each. Similarly there are three 'generations' of quarks, and each generation contains two quarks. The three generations are the up and down quarks, the charmed and strange quarks, and the top and bottom quarks. All the quarks and leptons are spin-$\frac{1}{2}$ fermions. The only difference between the three particle generations in the standard model is their masses. The reason for this seemingly redundant replication is completely unknown at present.

The quarks have the additional property that each comes in three 'colours' – red, green, and blue[3]. It is this colour degree of freedom that the $SU(3)$ group acts on, which is responsible for the strong nuclear force. The $SU(3)$ gauge bosons are called gluons. The restriction of the standard model to quarks and gluons is often referred to as **quantum chromodynamics** (QCD).

The standard model Lagrangian contains no mass terms for any of the fermions. This may sound strange, but we will see later that masses are generated by spontaneous symmetry breaking. For such *massless* Dirac particles, the Dirac equation actually separates into independent equations for the positive and negative chirality (right and left handed) parts. This means we can place the left and right-handed components of a particle into different representations of the gauge group, meaning they respond differently to the fundamental forces! In fact, nature forces us to do this, because the

[3]These are just fanciful names; elementary particles can not be said to have a colour in the traditional sense.

weak nuclear force only affects left-handed particles. For this reason, the standard model is said to be a **chiral** theory.

To emphasise the chiral nature of the theory, and that the $U(1)$ part of the gauge group represents hypercharge rather than electromagnetism, the standard model group is often written as $SU(3)_c \times SU(2)_L \times U(1)_Y$, where c stands for colour, L for left-handed, and Y for hypercharge[4].

The representations of $SU(3)_c \times SU(2)_L \times U(1)_Y$ are usually given as follows. For some field ψ, we write $\psi \sim (\mathbf{n_1}, \mathbf{n_2}, c)$ where n_1 is the dimension of its $SU(3)$ representation, n_2 is the dimension of its $SU(2)$ representation[5] , and c is its hypercharge (analogous to ordinary electric charge).

We will give the representations for the first generation of particles. As mentioned above, the other two generations are identical. The notation is as follows: $f_L = (\nu_e \quad e)_L^T$ is the left-handed electron and its neutrino, written explicitly as an $SU(2)$ doublet. Similarly $Q_L = (u \quad d)_L^T$ is the left-handed up and down quarks. e_R is the right-handed electron; u_R and d_R are the right-handed up and down quarks. The corresponding representations are,

$$\begin{aligned} f_L &\sim (\mathbf{1}, \mathbf{2}, -1) & Q_L &\sim (\mathbf{3}, \mathbf{2}, +1/3) \\ u_R &\sim (\mathbf{3}, \mathbf{1}, +4/3) & d_R &\sim (\mathbf{3}, \mathbf{1}, -2/3) \\ & & e_R \sim (\mathbf{1}, \mathbf{1}, -2) & \end{aligned} \tag{1}$$

We can see now that we can't include ordinary mass terms for the fermions, because they would not be gauge-invariant. For example, an electron mass term would be $m_e(\bar{e}_L e_R + \bar{e}_R e_L)$, and this is not invariant under $SU(2)_L \times U(1)_Y$.

Electromagnetism is incorporated in the standard model in a somewhat peculiar way. If we denote the generator of the hypercharge group by Y (such that, for example, $Y \cdot u_R = \frac{4}{3} u_R$), and the diagonal generator of $SU(2)_L$ by I_3, then it is easily verified from the above representations and the known electric charges of the particles that the electric charge operator Q is given by,

$$Q = I_3 + \frac{Y}{2}$$

Therefore the electromagnetic gauge group $U(1)_Q$ is contained in $SU(2)_L \times U(1)_Y$, and the photon field is a linear combination of the gauge fields of these groups.

So far there are a few problems with what we have presented. Gauge invariance precludes realistic fermion masses, and demands that the weak gauge bosons are massless, whereas we know this is not the case in nature. These issues will be resolved after we discuss spontaneous symmetry breaking.

2.2.1 Confinement in QCD

An important fact concerning the standard model is that quarks and gluons have never been observed on their own. Instead, all particles measured at particle accelerators are singlets under the colour $SU(3)$ group. This is a property known as **confinement** — it seems that coloured particles cannot exist freely, but instead are 'confined' inside composite colourless particles.

Can QCD explain confinement? All indications are that it can, but nobody has managed to prove it. It *is* true that the strength of the strong interaction is predicted by QCD to decrease at high energies, to the point that quarks barely interact at all if pushed close enough together[6] (remember the inverse relationship between energy and distance). This is known as **asymptotic freedom**. Conversely, if the quarks begin to separate, the strength of their interaction must increase.

Unfortunately though, as distances increase, the strength of the QCD interaction increases to the point where the effective coupling constant becomes larger than 1. Beyond this point, perturbative QFT is useless, so if confinement is part of QCD, it is a non-perturbative effect. Numerical simulations of the non-perturbative regime indicate that confinement does indeed occur, but this can not be considered absolute proof.

The energy scale at which the non-perturbative effects become important is called the **confinement scale**, and is denoted by $\Lambda_{QCD} \simeq 200MeV$. It is expected therefore that the size of composite particles made of quarks should be approximately Λ_{QCD}^{-1}. Indeed, this gives a length scale $\simeq 10^{-15}m$, which is roughly the size of the proton.

Closely related to confinement is another postulated feature of QCD referred to as the mass gap. A theory is said to exhibit a **mass gap** if there is a non-zero lower bound to the masses of physical states in the theory. It is believed that QCD has a mass gap because of confinement. If coloured

[4] The fact that Y stands for hypercharge can be considered a quantum spelling anomaly.

[5] $\mathbf{n}^*$ is used if the representation corresponds to the complex conjugate of the 'usual' n-dimensional representation.

[6] We are talking about the strong force only here; the electromagnetic interaction is strong at short distances.

particles are confined by the strong force, there is an associated energy that should be of order Λ_{QCD}. This manifests as the mass of the composite particle.

It is important to realise that because QCD has a non-Abelian gauge group, the gluons carry colour, and as such, because of confinement, cannot propagate freely. It is believed that massive bound states of gluons exist, called glueballs, although these have not been conclusively detected yet.

2.3 Spontaneous symmetry breaking

To understand spontaneous symmetry breaking, we first need to recognise a general feature of perturbative quantum field theory. In constructing Feynman diagrams, and the corresponding amplitudes, directly from the Lagrangian, we are implicitly expanding around the point in Hilbert space where all fields have vanishing expectation value. This is only legitimate if the corresponding point in classical field space is stable. Intuitively this is obvious; canonical quantisation amounts to 'quantising' the oscillations around a classical stable point. At an unstable point, some perturbations will not result in oscillatory behaviour, but instead will grow exponentially. For a detailed discussion of these issues, see [5]. In general, there will be some set of points in field space that correspond to the global minimum of energy. This set is known as the **vacuum manifold**[7], for any such point constitutes a possible quantum vacuum state. Ordinarily, the vacuum manifold consists of the single point where all fields vanish. In the case of spontaneous symmetry breaking, this is not the case.

Suppose we have a theory with gauge group G, and a scalar field ϕ in some non-trivial representation of G. We will see from examples that it is easy to write down a gauge-invariant potential $V(\phi)$ such that the point $\phi = 0$ is unstable. The vacuum manifold will then be some set of points where $\phi \neq 0$. Choose some point p from this set to correspond to the quantum vacuum. ϕ is in an irreducible representation, so the group $H \leq G$ which leaves p invariant will be a proper subgroup. Therefore if the vacuum of our universe corresponds to the point p, we will only observe the *smaller* symmetry described by H. We say that the symmetry has been broken from G to H.

A fact that is often useful is that if we write $g = exp(-iT)$, where T is in the Lie algebra of G, then g fixes the point p if and only if $T \cdot p = 0$. Thus a subgroup is unbroken if and only if its generators annihilate the vacuum. Often the terms broken and unbroken generators are used.

It is important to remember that the symmetry of the full theory is still G. It is the *vacuum* that has a smaller symmetry group, and it is for this reason that only a reduced symmetry would be directly observed in nature. The term **hidden symmetry** is sometimes used in this context.

Consider an element $g \in G$ that does not fix p. Because the potential is G-invariant, the point $g \cdot p$ must have the same energy as p. In other words, it is another point of the vacuum manifold. In this way, the vacuum manifold can be generated by the 'broken' elements of G, and we can see that topologically, it is the coset space G/H.

We will see more features of spontaneous symmetry breaking when we discuss the particular breaking that occurs in the standard model.

2.3.1 Breaking the electroweak symmetry of the standard model.

In the case of the standard model, we need to break the symmetry to $SU(3) \times U(1)_Q$, which is the unbroken gauge symmetry observed in nature. To do this we introduce the **Higgs field** ϕ, which is a complex scalar field in the representation $(\mathbf{1}, \mathbf{2}, +1)$. It is important that ϕ be a singlet under $SU(3)$, because this guarantees that colour symmetry will not be broken. ϕ is an $SU(2)$ doublet, so we write it in terms of its four real components as,

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

We need to introduce a gauge-invariant potential such that $\phi = 0$ is unstable. Explicitly, the Higgs Lagrangian is taken to be,

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= (D_\mu \phi)^\dagger D^\mu \phi - V(\phi) \\ &= (D_\mu \phi)^\dagger D^\mu \phi - \lambda(\phi^\dagger \phi - v^2)^2 \end{aligned}$$

With $\lambda > 0$, we can see that the classical minimum of the potential occurs when $|\phi| = v$, whereas $\phi = 0$ sits on top of a potential 'hill'. Therefore any stable classical solution must have $|\phi| = v$. In

[7] In all cases of interest, it will be obvious that this is indeed a manifold.

the quantum theory, this means that the Higgs field has a non-zero **vacuum expectation value** (VEV). We can perform an $SU(2) \times U(1)$ transformation such that,

$$\phi_0 \equiv \langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

We have achieved the required symmetry breaking! To verify this, recall that $Q = I_3 + \frac{Y}{2}$. It follows that $Q \cdot \phi_0 = 0$, and this means the electromagnetic gauge group has remained unbroken. It is easy to check that no other generators of $SU(2) \times U(1)$ annihilate the above VEV.

It is perturbations around ϕ_0 that appear in the quantised theory, and thus correspond to the particles we might actually detect. As such, we re-write the Higgs field as,

$$\phi = \begin{pmatrix} (\phi_1 + i\phi_2)/\sqrt{2} \\ v + (\sigma + i\phi_4)/\sqrt{2} \end{pmatrix}$$

Re-writing the Lagrangian in terms of σ instead of ϕ_3, we see something interesting; there are now mass terms for the gauge bosons corresponding to the broken generators. This is a generic feature of spontaneous symmetry breaking. As might be guessed, a mass term for a vector particle is $m^2 A_\mu A^\mu$; it can be verified that the masses generated by symmetry breaking are proportional to v.

Spontaneous symmetry breaking using the Higgs field generates the required masses for the W and Z bosons, and detailed calculations even yield the experimentally observed ratio of these masses. This is an extraordinary achievement of the theory[8]. Note that the photon remains massless, as the electromagnetic $U(1)$ group is unbroken.

2.3.2 Fermion masses

We have seen that spontaneous symmetry breaking generates the required masses for the weak gauge bosons, and it turns out that it can also generate masses for the standard model fermions. For example, the following gauge-invariant term is added to the standard model Lagrangian,

$$\lambda_e \overline{f_L} \phi e_R + \text{h.c.}$$

where λ_e is a dimensionless constant, and h.c. stands for 'Hermitian conjugate'. After spontaneous symmetry breaking, this term splits into two parts, one of which is $\lambda_e v \overline{e_L} e_R +$ h.c. This is just a mass term for the electron, corresponding to mass $\lambda_e v$. Masses can be generated for the quarks in the same way (the complex conjugate ϕ^c of the Higgs field must be used for the up quark mass term). The Yukawa coupling constants, such as λ_e, are chosen to produce the experimentally measured masses. Thus the standard model does not predict the masses of fermions; they must be put in as parameters.

3 Grand Unified Theories

The standard model is in excellent agreement with a large range of experiments. Even so, we know it cannot be the ultimate theory of nature. For example, it contains no explanation for the existence of dark matter, and does not deal with gravity at all. So we know that we must eventually go beyond the standard model if we are to understand how the universe works at the most fundamental level. This motivates the study of theories which extend on the standard model, sometimes simplifying certain aspects of it at the same time.

One possibility along these lines is to consider a gauge theory with a single simple gauge group (as opposed to the product of three such groups that the standard model uses), which contains the standard model as a subgroup. This means, for example, that there is only one gauge coupling constant, instead of three. It also means we can fit more particles into a single representation of the group (see below). Such a theory is referred to as a 'grand unified theory' (GUT). The two most commonly considered unification groups are $SU(5)$ and $SO(10)$.

[8] The Z boson had not even been discovered experimentally when electroweak theory was first proposed.

3.1 $SU(5)$ unification

The most important GUT for this project is the one with gauge group $SU(5)$. The standard model gauge group is contained in $SU(5)$ via the embedding $\varphi : SU(3) \times SU(2) \times U(1) \to SU(5)$ given by:

$$\varphi(V, U, e^{i\theta}) = \begin{pmatrix} \begin{pmatrix} & V & \end{pmatrix} & \\ & \begin{pmatrix} U \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} e^{2\theta} & & & & \\ & e^{2\theta} & & & \\ & & e^{2\theta} & & \\ & & & e^{-3\theta} & \\ & & & & e^{-3\theta} \end{pmatrix}$$

We need to put the standard model fermions into representations of $SU(5)$ such that the induced representations of $SU(3) \times SU(2) \times U(1)$ are those of the standard model. It turns out that two irreducible representations of $SU(5)$ – the $\mathbf{5}^*$ and the $\mathbf{10}$ – are enough to contain all the standard model fermions in this way. The $\mathbf{5}^*$ is the conjugate of the fundamental representation, and the $\mathbf{10}$ is the anti-symmetric rank 2 tensor. The standard model particles are assigned to these representations as follows [4]:

$$\mathbf{5}^* : \left((d_{rR})^c \ \ (d_{gR})^c \ \ (d_{bR})^c \ \ e_L \ \ \nu_{eL} \right)$$

$$\mathbf{10} : \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & (u_{bR})^c & (-u_{gR})^c & u_{rL} & d_{rL} \\ (-u_{bR})^c & 0 & (u_{rR})^c & u_{gL} & d_{gL} \\ (u_{gR})^c & (-u_{rR})^c & 0 & u_{bL} & d_{bL} \\ -u_{rL} & -u_{bL} & -u_{gL} & 0 & e_L^+ \\ -d_{rL} & -d_{bL} & -d_{gL} & -e_L^+ & 0 \end{pmatrix}$$

where subscripts L, R stand for left- and right-handed, r, g, b stand for red, green, and blue, and a superscript c indicates the charge-conjugate wavefunction, $\psi^c = i\gamma^2\psi^*$. Note that charge conjugation also changes the chirality. It should at least be obvious that the fields in the $\mathbf{5}^*$ transform as they should under $SU(3) \times SU(2) \times U(1)$.

What about the gauge bosons of the theory? As always, they are in the adjoint representation of the gauge group, which in the case of $SU(5)$ is 24-dimensional. This breaks down under $SU(3) \times SU(2) \times U(1)$ as,

$$\mathbf{24} \longrightarrow (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \mathbf{2}, -\frac{5}{3}) \oplus (\mathbf{3}^*, \mathbf{2}, \frac{5}{3})$$

We can identify the first three representations here with the gluons, the W bosons, and the B boson respectively. The last two correspond to the other $SU(5)$ gauge bosons, usually referred to generically as the X gauge bosons.

The X gauge bosons lead to a richer phenomenology than is exhibited by the standard model. For example, they mediate processes in which quarks change into leptons, and this makes the proton unstable. This is not catastrophic for the GUT idea, because experiments only set upper bounds on the proton lifetime. There are ongoing experiments to search for evidence of proton decay which are motivated by, among other things, theories such as the $SU(5)$ model just presented.

3.1.1 Symmetry breaking in the $SU(5)$ model

The unified picture provided by $SU(5)$ is theoretically attractive, but we don't observe $SU(5)$ symmetry in nature, so if it exists, it must be spontaneously broken. The easiest way to break the symmetry to the standard model group is to introduce a scalar field in the adjoint representation of $SU(5)$. With an appropriately chosen potential, the vacuum expectation value of such a field breaks the symmetry to that of the standard model.

For a realistic description of the world, the usual electroweak symmetry breaking also has to be incorporated in the model. This can be done with a Higgs field in the $\mathbf{5}$ representation, which contains a $(\mathbf{1}, \mathbf{2}, +1)$ under the standard model group.

The symmetry breaking must be arranged so that $SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$ happens at a very high energy scale. This gives correspondingly large masses to the X bosons, which explains why they haven't been observed. The second stage of symmetry breaking occurs at the electroweak scale of the standard model. The disparity between the two mass scales is unnatural, and maintaining it requires delicate fine-tuning of the mass parameters in the theory. This is referred to as the **gauge hierarchy problem**.

4 Extra Dimensions and Branes

In recent years, many theories of particle physics have incorporated extra spacetime dimensions in one way or another. String theory and M theory require 10 or 11 dimensions for a consistent formulation, while numerous phenomenological models exist with one or more extra spatial dimensions. Here we review the aspects of these models relevant to this project. For extensive reviews, see [6], [7], and for a detailed analysis of the phenomenology of such models, see [8].

4.1 Kaluza-Klein theory

The first time that extra spacetime dimensions were seriously suggested in a physical theory was in the 1920s, in the context of what is now called Kaluza-Klein theory. Following Einstein's successful description of gravity in terms of spacetime geometry, it was thought that perhaps electromagnetism (the only other force known at the time) could be described similarly. Such a model was put forth by Theodor Kaluza. The crucial assumption of the theory is that spacetime has one extra spatial dimension. If spacetime transformations are restricted to be four-dimensional, so that $x^\mu \longrightarrow x'^\mu(x^\mu),\ \mu = 0, 1, 2, 3,\ x^4 \longrightarrow x'^4 = x^4$, then the entries $g_{\mu\nu},\ \mu, \nu = 0, 1, 2, 3$ in the five-dimensional metric behave like a four-dimensional metric. The entries $g_{\mu 4}$ transform like an ordinary four-vector, which is taken to be the electromagnetic potential A_μ. Under certain assumptions about the metric, electromagnetic gauge transformations can be realised by coordinate changes $x'^\mu = x^\mu,\ x'^4 = x^4 + f(x^\mu)$.

Taken at face value, the theory seems unrealistic, because it requires an extra spatial dimension. But it was soon realised that this is not an insurmountable problem. The argument is that if the fifth dimension was described by a circle (so that the topology of the universe is $\mathbb{R}^4 \times S^1$), then it may not be observable if the radius R of that circle is very small. On scales significantly larger than R, the three-dimensional Gauss' law would still hold, for example, because at distance R from the source, the flux would fill the fourth spatial dimension, and beyond that only dilute in the three large dimensions. We say that the extra dimension is 'compactified on a circle'. This line of reasoning is credited to Oskar Klein, and 'Kaluza-Klein theory' usually refers to the five-dimensional theory with the extra dimension compactified as above.

4.2 Dimensional reduction

Let's examine the consequences of the Kaluza-Klein extra dimension in the context of field theory. For simplicity we will consider a free real scalar field of mass m (we really mean that the spectrum of the quantised theory consists of particles of mass m), described by the Klein-Gordon action:

$$\mathcal{S} = \frac{1}{2}\int d^4x \int dy\ (\partial_M \Phi \partial^M \Phi - m^2 \Phi^2)$$

where y labels the extra dimension, and M is a five dimensional Lorentz index. This describes a theory living in all five dimensions, but if R is very small, we will only be aware of four dimensions. Obtaining the effective four-dimensional theory from the full theory is known as 'dimensional reduction', and proceeds as follows:

Isolate the extra-dimensional dependence of the field by writing it as a Fourier series in y:

$$\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}}\phi_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} \left[\phi_n(x^\mu)\cos\left(\frac{n}{R}y\right) + \phi'_n(x^\mu)\sin\left(\frac{n}{R}y\right)\right]$$

Notice that all y-dependence of Φ is now contained in the *fixed* trigonometric functions. Performing field variations amounts to individually varying each ϕ and ϕ'. As such, we can perform the y-integral explicitly without losing any information from our theory. Here we simply quote the result, which is easily obtained:

$$\begin{aligned}\mathcal{S} = \frac{1}{2}\int d^4x \Bigg\{ \partial_\mu\phi_0\partial^\mu\phi_0 - m^2\phi_0^2 \quad &+ \quad \sum_{n=1}^{\infty}\left[\partial_\mu\phi_n\partial^\mu\phi_n - \left(m^2 + \frac{n^2}{R^2}\right)\phi_n^2\right] \\ &+ \quad \sum_{n=1}^{\infty}\left[\partial_\mu\phi'_n\partial^\mu\phi'_n - \left(m^2 + \frac{n^2}{R^2}\right)\phi_n'^2\right]\Bigg\}\end{aligned}$$

This now has a very straightforward interpretation. The first two terms describe a four-dimensional scalar with the same mass as the five-dimensional particle. The two infinite summations describe a doubly-degenerate spectrum of scalars with mass-squared of $m_n^2 = m^2 + \frac{n^2}{R^2}$. So excitation of the higher modes to which these correspond would appear in four dimensions to be the creation of particles of mass m_n. If R is very small, these masses will be extremely large even for $n = 1$, so in most circumstances we will only observe the 'zero mode' ϕ_0.

An equivalent approach, which is applicable to more complicated extra-dimensional models, is to write:

$$\Phi(x^\mu, y) = \sum_{n=0}^{\infty} \eta_n(y)\phi_n(x^\mu)$$

and demand that each ϕ_n is a 4D scalar field, so that $\partial^\mu\partial_\mu\phi_n + m_n^2\phi_n = 0$. This leads to an equation for η_n depending on m_n. This is treated as an eigenvalue equation, with eigenvalues m_n and eigenfunctions η_n, and the spectrum of the 4D theory is derived in this way. Essentially, what we did above was 'guess' the right form of the functions η_n.

4.3 Large extra dimensions

More recently, the Kaluza-Klein idea has been extended to include the possibility of multiple, reasonably large extra dimensions [9]. The main motivation for this is to try to solve the 'hierarchy problem', which refers to the fact that standard model parameters have to be fine-tuned to approximately one part in 10^{17} [10] to ensure the electroweak mass scale[9] m_{EW} is as low as we observe. The problem arises because the fundamental Planck mass M_{Pl} is so large compared to m_{EW}. In natural units, Newton's gravitational constant G is given by $G = \frac{1}{M_{PL}^2}$, so the question over this hierarchy is sometimes phrased as "why is gravity so weak?"

Consider the gravitational field of a point mass m. In a four dimensional universe the strength g of this field at a distance r is given by:

$$g = \frac{1}{M_{PL}^2}\frac{m}{r^2}$$

Now suppose our universe has N extra dimensions, and a fundamental mass scale M^*. Then we would expect the above to be replaced by,

$$g = \frac{1}{M^{*2+N}}\frac{m}{r^{2+N}}$$

But if the N extra dimensions are each compact, say of 'size' R, then when $r > R$, they become saturated - the flux can dilute no further in those dimensions. Of course, it spreads out as normal in the three large dimensions, so if R^N is used as the volume of the extra dimensions, the field strength at $r > R$ is,

$$g = \frac{1}{M^{*2+N}R^N}\frac{m}{r^2}$$

Comparing this with the familiar four-dimensional law, we see that,

$$M_{Pl}^2 = M^{*2+N}R^N$$

If R is big enough, we can have $M^* \sim m_{EW}$ and still produce a huge four-dimensional Planck mass, thus solving the hierarchy problem.

4.3.1 Branes

Of course, if the extra dimensions are 'large' as suggested above, we have to explain why they haven't been observed in particle physics experiments, which probe very small length scales. One approach is to notice that gravitational experiments alone put quite weak bounds on the size of extra dimensions; the $\frac{1}{r^2}$ form of Newton's law has only been tested down to scales of about .2mm. We can postulate that the extra dimensions are of order $\lesssim$ .1mm, but that all fields except gravity are confined to a (3+1)-dimensional submanifold of the full spacetime. This is dubbed a 'brane' —short for membrane. Another possibility is explored in [11]. Roughly speaking, gravity is trapped on one brane, and standard model fields on another. The effective $(3+1)$-dimensional masses of the standard model fields are then exponentially smaller than the mass parameters in the $(4+1)$-dimensional theory, and this explains the smallness of m_{EW}. In theories with branes, the full spacetime is usually referred to as the 'bulk'.

Putting such infinitely thin branes into the theory 'by hand' is unattractive from a field-theoretical point of view, so it is desirable to develop mechanisms by which fields can be dynamically localised in brane-like configurations. This is the focus of this project, and we will have more to say about it soon.

[9]This can be taken to be eg. the mass of the W boson.

4.4 Non-compact extra dimensions

In the previous sections we argued that extra dimensions have to be compactified to yield an effective four-dimensional theory. We made one crucial assumption though, without stating so explicitly - we assumed that the background metric was that of five-dimensional Minkowski space. In recent years it has been shown that if this assumption is dropped, an effective four-dimensional theory can be obtained from a five-dimensional theory with an *infinite* extra dimension.

In their paper [12], Randall and Sundrum demanded only four-dimensional Poincaré invariance of the universe, by taking as a metric ansatz,

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

$e^{-2\sigma(y)}$ is usually referred to as the 'warp factor', and the solution found is such that it decreases rapidly away from $y = 0$. All fields other than gravity are placed on a brane at $y = 0$. Note that this theory has a very different structure to anything we've discussed so far. In particular, if spacetime was a product $X \times Y$ of two (possibly pseudo-)Riemannian manifolds X and Y, with metrics $ds_X^2 = g_{X\mu\nu}dx^\mu dx^\nu$ and $ds_Y^2 = g_{Y\mu\nu}dy^\mu dy^\nu$, then the spacetime metric would be $ds^2 = ds_X^2 + ds_Y^2$. This is not the case in the Randall-Sundrum model, because the $(3+1)$-dimensional part of the metric depends on the extra dimensional coordinate.

One technical point of interest is that the existence of the solution found depends on a fine-tuning between the five-dimensional cosmological constant and the energy density of the brane, called the **brane tension**.

When fluctuations around the above metric are considered[10], a sensible four-dimensional action can be obtained by dimensional reduction. The spectrum of this theory is qualitatively different to the Kaluza-Klein case though. Again, there is a single zero mode, which corresponds to a massless four-dimensional graviton, but the fact that the extra dimension is infinite changes the structure of the excited modes. In fact, where before the four-dimensional masses formed a 'tower' of discrete values, here we get a continuum of values, starting at $m = 0$. Usually this would spell trouble for the effective four-dimensional theory (an infinite number of particles of slightly different masses, but otherwise identical), but in the warped background, it turns out not to be a problem. The warp factor means that the continuum modes make an insignificant contribution to four-dimensional physics. In fact, the usual Newtonian potential is replaced by,

$$V(r) = G\frac{m_1 m_2}{r}\left(1 + \frac{1}{r^2k^2}\right)$$

where k is a very large energy (remember in natural units energy is the same as inverse length). The second term is the correction due to continuum modes, and is only significant for very small values of r – smaller than those at which the $\frac{1}{r}$ behaviour has been tested. The detailed analysis leading to the above conclusions can be found in [12].

4.5 Dynamical localisation

The Randall-Sundrum model realises localisation of gravity, yielding an effective $(3+1)$-dimensional theory in the context of an infinite extra dimension. We now need to discuss field-theoretic methods by which the standard model fields can also be localised in such a model, instead of being artificially confined to a 'brane'.

4.5.1 Fermion localisation by a kink

There is a well-known mechanism for confining fermions to a brane-like object, which goes back to work of Jackiw and Rebbi [13], and was originally used in the current context by Rubakov and Shaposhnikov [14].

We start with a theory containing a single scalar field Φ in 4+1 Minkowski spacetime. We demand that the theory be symmetric under the $\mathbb{Z}_2$ transformation $\Phi \longrightarrow -\Phi$. The resulting Lagrangian is,

$$\mathcal{L} = \frac{1}{2}\partial^M\Phi\partial_M\Phi - \lambda(\Phi^2 - v^2)^2$$

[10]When quantised, these modes yield gravitons. Although there is no accepted theory of quantum gravity yet, Newton's law *can* be derived from an exchange of gravitons in Minkowski space, so this is a sensible approach to studying the effective four-dimensional gravity in the theory.

where λ and v are positive constants. The sign of the Φ^2 term has been chosen so that the $\mathbb{Z}_2$ symmetry is spontaneously broken. The equation of motion resulting from the above is,

$$\partial_M \partial^M \Phi + 4\lambda\Phi(\Phi^2 - v^2) = 0$$

This equation has stable solutions $\Phi = \pm v$, analogous to the symmetry breaking in the Higgs model discussed earlier. In this instance though, we are interested in a more complicated possibility. If we consider the case where Φ depends only on y, we find the solution,

$$\Phi = v \tanh(ay), \quad a = \sqrt{2\lambda} v$$

This solution is well known, and is called a **kink**, because of the shape of its graph. Notice that as $y \longrightarrow \pm\infty$, $\Phi \longrightarrow \pm v$; the field interpolates between its two vacua. It is an example of a topological defect called a domain wall, and we will try to identify the centre of this wall ($|y| \simeq 0$) with our four-dimensional universe.

We now introduce a fermion Ψ into the theory. The Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}\partial^M \Phi \partial_M \Phi - \lambda(\Phi^2 - v^2)^2 + i\overline{\Psi}\Gamma^M \partial_M \Psi + h\Phi\overline{\Psi}\Psi$$

The gamma matrices Γ^M can be taken as $\Gamma^\mu = \gamma^\mu$, $\Gamma^4 = -i\gamma^5$, where the γ's are the familiar Dirac matrices. h is a constant. We take a kink-like solution for Φ that interpolates between $\pm v$, and look for the corresponding form of Ψ. The Dirac equation that follows from the above is,

$$i\Gamma^M \partial_M \Psi + h\Phi\Psi = 0$$

We look for a localised zero-mode by writing $\Psi(x^\mu, y) = f(y)\psi(x^\mu)$, and requiring ψ to satisfy the massless 4D Dirac equation $\gamma^\mu \partial_\mu \psi = 0$. Then the above reduces to:

$$\frac{df}{dy}\gamma^5 \psi + h\Phi f \psi = 0$$

From this we see that the solution will depend on which chirality we take for the four-dimensional fermion ψ. Let $\gamma^5 \psi = \epsilon\psi$, $\epsilon = \pm 1$. Then we can cancel ψ from the above equation, and find the solution for f:

$$f(y) \propto e^{-\epsilon h \int dy \Phi(y)}$$

For a right-handed fermion $\epsilon = 1$, and $f(y)$ decays exponentially as $|y| \to \infty$. We can substitute this solution back into the action and perform the y-integral, thereby obtaining a four dimensional action for ψ. So right-handed fermions are localised in the domain wall[11].

4.5.2 Gauge bosons

Of course, fermions are not the only dynamical fields in the standard model. There are also gauge bosons, which are spin-1 particles. There have been several suggestions for mechanisms by which they can be localised, but none yet that have been widely accepted. Here we introduce two that are relevant to this project.

Dilaton-mediated localisation

Kehagias and Tamvakis suggested a model in which photons are localised in a Randall-Sundrum-like background via coupling to a scalar field called a dilaton [2]. In the photon kinetic term, $F_{\mu\nu}F^{\mu\nu}$ is replaced by $e^{-K\pi}F_{\mu\nu}F^{\mu\nu}$, where π is the dilaton. Analogously to what we did above with the kink, a solution is found for π by ignoring the coupling to the photon. This solution allows dimensional reduction of the photon action, yielding the usual four-dimensional theory, plus corrections from massive Kaluza-Klein modes (which actually have a continuum of mass values, because the extra dimension is infinite).

[11] We can choose to localise left-handed fermions instead, by changing the sign of h, or taking the alternative solution $\Phi(y) = -v \tanh(ay)$.

The Dvali-Shifman mechanism
A completely different approach is given by Dvali and Shifman in [1]. The idea relies crucially on having a gauge theory that exhibits the inherently quantum-mechanical phenomenon of confinement[12], and an associated mass gap. As an example, they present a toy $SU(2)$ model. Two scalar fields are employed such that $SU(2)$ is broken to $U(1)$ on a domain wall. They argue that in the bulk, the $SU(2)$ symmetry is unbroken, so the theory should exhibit confinement, with some characteristic mass scale Λ. Because $U(1)$ is unbroken, its gauge boson remains massless, and thus at energy scales less than Λ, should remain trapped on the wall.

Note that this is a (3+1)-dimensional model, with a (2+1)-dimensional domain wall. It generalises to any number of dimensions, but for now we'll stick to $3 + 1$. The theory contains a scalar field χ in the adjoint (triplet) representation of $SU(2)$, and another scalar η which is uncharged under $SU(2)$[13]. The Lagrangian is,

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{a\mu\nu} + Tr\left[(D_\mu\chi)^\dagger D^\mu \chi\right] + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\lambda'}{2}(Tr\left[\chi^2\right] + \kappa^2 - v^2 + \eta^2)^2 - \lambda(\eta^2 - v^2)^2$$

where $G^a_{\mu\nu}$ is the field strength tensor, g is the gauge coupling constant, v and κ are mass parameters taken to be much larger than the confinement scale Λ of the $SU(2)$ theory, and λ, λ' are positive dimensionless constants.

The argument proceeds as follows. The classical minimum of the above potential will have $\eta \simeq v$ or $-v$, and $\chi = 0$. The $SU(2)$ symmetry is unbroken, because η is a singlet under $SU(2)$. But if we ignore initially the coupling to χ, we know that η has a static kink solution, interpolating between v and $-v$ at eg. $y = \pm\infty$. Thus a domain wall is formed around the plane where $\eta = 0$. We expect a similar solution to persist when the coupling to χ is turned on.

Consider the potential inside the domain wall. If $v^2 > \kappa^2$, then when $\eta \simeq 0$, $\chi = 0$ is longer a minimum. Hence we expect χ to develop a non-zero value inside the wall, breaking $SU(2)$ to $U(1)$.

We can be a little more rigorous than this, following the reasoning originally presented in [15]. The equations of motion are satisfied if χ is identically zero and $\eta(y) = v\tanh(my)$ is a kink, where $m = \sqrt{2\lambda}v$. But a sensible quantum theory can only be built on a *stable* classical solution [5]. To investigate stability, we consider a perturbation of the third component of χ. Let $\chi_3 = \epsilon\tilde{\chi}e^{-i\omega t}$, where we assume that $\tilde{\chi}$ is a function of y only. Keeping only terms of first order in ϵ, the equation for $\tilde{\chi}$ is:

$$\left[-\frac{\partial^2}{\partial y^2} + \frac{\lambda'}{2}\Big(\kappa^2 + v^2(\tanh^2(my) - 1)\Big)\right]\tilde{\chi} = \omega^2\tilde{\chi}$$

Notice that this is just a time-independent Schrödinger equation. With $\kappa = 0$, the potential is negative definite, and there is known to be a negative-energy bound state solution. By continuity, such a solution exists for some range of non-zero κ. But the 'energy' here is ω^2, so negative energy means ω is imaginary, and thus our perturbation grows exponentially with time. Therefore the solution we started with is unstable, and the stable classical solution must have non-zero χ.

This set-up is based on earlier work done by Witten [15] in the context of cosmic strings.

5 Fields in Curved Spacetime

In light of the successes of the dynamical localisation paradigm, we are in a position to consider constructing a 5D model which yields an effective 4D theory for gravity, fermions, and gauge bosons, all dynamically localised. We immediately encounter a new complication.

The standard treatments of field theory apply only in Minkowski spacetime. This is usually a good approximation, because only in exceptional circumstances is spacetime curvature large enough to be important. Unfortunately, the spacetime of Randall and Sundrum is inherently curved, so we must address the issue of how to describe fields in a curved background spacetime. This would be relatively straightforward but for the existence of fermions. The reason is that bosons are described by tensor fields, which are easily handled on any manifold, but fermions are described by spinor fields. The incorporation of such fields into a curved background requires a slightly different formulation of general relativity, which has the added advantage of demonstrating that it can be considered as a gauge theory, with gauge group the Lorentz group. Therefore this initially annoying diversion turns out to be quite interesting in its own right.

[12] Earlier we discussed confinement in the case of QCD. In fact it should be a feature of a general class of non-Abelian gauge theories, including the presente one, and the $SU(5)$ theory presented later.

[13] [1] actually has two fermion doublets as well, but we ignore these here as they don't affect the features we are interested in.

5.1 Gravity as a gauge theory

This section does not constitute a review of general relativity. It merely presents a different approach to general relativity to those who are already familiar with the theory. This is necessary for describing fermions in a curved spacetime, which don't fall under the usual formalism as spinor fields are non-tensorial. For generality we work in an $(n+1)$-dimensional spacetime.

The physical principles behind general relativity can by summarised by the following two assumptions:

1. Gravity is the manifestation of the fact that spacetime is not flat. The presence of matter changes the geometry of spacetime, and free test particles follow geodesics in this geometry.
2. For a free-falling observer, the laws of special relativity hold locally.

A reference frame in which a free-falling observer is at rest will hereafter be referred to as a 'locally inertial' frame, to remind us of point 2. We can translate these two physical principles into mathematical statements:

1. The effects of gravity are encapsulated in the spacetime metric $g_{\mu\nu}$.
2. At each spacetime point X, there exists a reference frame in which:

$$\begin{aligned} g_{\mu\nu}(X) &= \eta_{\mu\nu} \\ \partial_\rho g_{\mu\nu}(X) &= 0 \end{aligned}$$

If we denote by $\{\xi^\alpha_X\}$ the coordinates for which statement 2 holds, then in a general coordinate system with coordinates $\{x^\mu\}$ we will get:

$$g_{\mu\nu}(X) = \frac{\partial \xi^\alpha_X}{\partial x^\mu}\frac{\partial \xi^\beta_X}{\partial x^\nu}\eta_{\alpha\beta} \tag{2}$$

Conversely, if (2) is satisfied, then $\{\xi^\alpha_X\}$ are locally inertial coordinates at X. Motivated by the above, we define an n-tuple of objects $\{V^0{}_\mu, \ldots, V^n{}_\mu\}$ by $V^\alpha{}_\mu(X) = \frac{\partial \xi^\alpha_X}{\partial x^\mu}$. Collectively, these are known as the vielbein (usually vierbein or tetrad in $3+1$ dimensions). We will now examine some of the properties of the vielbein.

First notice that if the coordinate system determined by $\{\xi^\alpha\}$ is locally inertial at X, then so is any coordinate system related to it by a Lorentz transformation. In other words, if Λ is any Lorentz transformation, then $\xi'^\alpha \equiv \Lambda^\alpha_\beta \xi^\beta$ will also satisfy the requirement (2). This is as we should expect – we postulated that special relativity should hold locally, and special relativity is Lorentz invariant.

So there are two distinct types of transformation we need to worry about: general coordinate transformations, which affect the coordinates on some open subset of the spacetime manifold, and local Lorentz transformations, which affect the locally inertial coordinates *at a point*.

It is easily checked that under a local Lorentz transformation Λ, the vielbein transforms according to the rule $V^\alpha_\mu \longrightarrow \Lambda^\alpha{}_\beta V^\beta{}_\mu$. So for fixed μ, V^α_μ transforms like a flat-space four-vector under local Lorentz transformations. We refer to α as a 'flat-space index', or simply 'flat index'.

Similarly, under a general coordinate transformation $x \longrightarrow x'(x)$, we get: $V^\alpha_\mu \longrightarrow \frac{\partial x^\nu}{\partial x'^\mu} V^\alpha{}_\nu$. So for fixed α, V^α_μ is simply a covariant vector field (a one-form). μ is referred to as a 'curved-space index', or 'curved index'.

By definition, we raise and lower flat indices with the flat-space metric η, and curved indices with the spacetime metric g. Using this we can easily prove the following useful results:

$$\begin{aligned} V^\alpha{}_\mu V_\beta{}^\mu = \delta^\alpha_\beta &\implies V_{\alpha\mu} V_\beta{}^\mu = \eta_{\alpha\beta} \\ V^\alpha{}_\mu V_\alpha{}^\nu = \delta^\nu_\mu &\implies V^\alpha{}_\mu V_{\alpha\nu} = g_{\mu\nu} \end{aligned}$$

The correct interpretation of the vielbein is that it represents a smooth choice of an 'orthonormal frame' (an orthonormal basis[14] for the tangent space) at each point. Given any vector field T, we can use the vielbein to obtain its components in the locally inertial coordinates:

$$\tilde{T}^\alpha \equiv V^\alpha{}_\mu T^\mu \qquad \tilde{T}_\alpha \equiv V_\alpha{}^\mu T_\mu$$

and similarly for tensor fields of any rank. These components are unaffected by general coordinate transformations, but transform 'as usual' under local Lorentz transformations.

How are we to incorporate derivatives into this framework? The reader may initially think that we have done away with such things as the Christoffel symbols, because in a locally inertial coordinate

[14]Because of the Lorentzian signature, we cannot find a basis for which $< V_\alpha, V_\beta >= \delta_{\alpha\beta}$, so we take 'orthonormal' to mean $< V_\alpha, V_\beta >= \eta_{\alpha\beta}$.

system, these vanish. This is too hasty however; the information usually carried by the Christoffel symbols is now encoded in the way the vielbein rotates as we move around in spacetime. In a general coordinate system, these details will be contained in the covariant derivatives of the vielbein itself. Because the vielbein constitutes a basis at each point, there exist co-efficients $\omega_{\alpha\beta\mu}$ (called **connection coefficients**) such that,

$$D_\mu V_\beta{}^\nu = \omega_{\alpha\beta\mu} V^{\alpha\nu}$$

This yields the formula which in practice is usually used to calculate the connection coefficients for a vielbein given in a particular coordinate system:

$$\omega_{\alpha\beta\mu} = V_\alpha{}^\nu D_\mu V_{\beta\nu}$$

It is easy to show from this that $\omega_{\alpha\beta\mu} = -\omega_{\beta\alpha\mu}$. This is not a coincidence. The generators of the Lorentz group are 4×4 matrices $a^\alpha{}_\beta$ satisfying $a_{\alpha\beta} = -a_{\beta\alpha}$ where $a_{\alpha\beta} \equiv \eta_{\alpha\gamma} a^\gamma{}_\beta$. In other words, each matrix ω_μ (with components $\omega^\alpha{}_{\beta\mu} \equiv \eta^{\alpha\gamma}\omega_{\gamma\beta\mu}$) belongs to the Lie algebra of the Lorentz group. We can choose a basis $\{E^{\alpha\beta}\}$ for the Lie algebra such that $\eta E^{\alpha\beta}$ has 1 in row-α, column-β, and -1 in row-β, column-α, with zeroes elsewhere[15]. Notice then that $E^{\alpha\beta} = -E^{\beta\alpha}$ (remember that each value of the indices denotes an entire matrix, *not* an entry in a matrix). For example,

$$\eta E^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\eta E^{10} \tag{3}$$

In this basis, we have,

$$\omega_\mu = \frac{1}{2}\omega_{\alpha\beta\mu} E^{\alpha\beta}$$

The factor of 1/2 comes about because we double count by including $E^{\alpha\beta}$ and $E^{\beta\alpha}$. We can now define a covariant derivative that transforms 'correctly' under local Lorentz transformations. For a vector field $T^\nu = V_\beta{}^\nu \tilde{T}^\beta$, the covariant derivative of its flat space components is given by,

$$\begin{aligned} D_\mu \tilde{T}^\beta &= \partial_\mu \tilde{T}^\beta + \omega^\beta{}_{\gamma\mu}\tilde{T}^\gamma \\ &= \partial_\mu \tilde{T}^\beta + \frac{1}{2}\omega_{\alpha\delta\mu}(E^{\alpha\delta})^\beta{}_\gamma \tilde{T}^\gamma \end{aligned}$$

where we have used the Leibniz rule, and $\omega^\beta{}_{\gamma\mu} \equiv \eta^{\beta\delta}\omega_{\delta\gamma\mu}$. This is a rather daunting array of indices, but if we suppress them, we see that we really just have the familiar gauge-covariant derivative:

$$D_\mu \tilde{T} = (\partial_\mu + \frac{1}{2}\omega_\mu \cdot E)\tilde{T}$$

Of course, we can use the vielbein to re-write this with all flat-space indices:

$$\begin{aligned} D_\alpha \tilde{T}^\beta &= V_\alpha{}^\mu D_\mu \tilde{T}^\beta \\ &= V_\alpha{}^\mu (\partial_\mu \tilde{T}^\beta + \frac{1}{2}\omega_\mu \cdot E^\beta{}_\delta T^\delta) \end{aligned}$$

It is easily verified that under a spacetime-dependent Lorentz transformation Λ, ω_μ transforms exactly as required by gauge theory:

$$\omega_\mu \longrightarrow \Lambda\omega_\mu\Lambda^{-1} + (\partial_\mu\Lambda)\Lambda^{-1}$$

In summary, a curved background spacetime can be incorporated into field theory by starting with the flat-space Lagrangian and making the replacement $\partial_\alpha \longrightarrow D_\alpha = V_\alpha{}^\mu D_\mu$ where D_μ will depend on which representation of the Lorentz group is being considered.

[15]The fact we have to multiply by η might be confusing. Really it just compensates for the Lorentzian nature of the spacetime metric.

5.1.1 Spinors and the spin connection

We can see that fermions are easily incorporated into this framework by following the familiar gauge theory prescription. The usual Dirac Lagrangian is invariant under (global) Lorentz transformations of Minkowski space. To incorporate gravity, we simply gauge this Lorentz symmetry ie. we replace $\partial_\mu \psi$ in the Dirac Lagrangian with some suitable 'covariant derivative' D_μ. Let's work out what D_μ should be.

From the theory of the Dirac equation, we know that under an infinitesimal Lorentz transformation given by $\Lambda^\gamma{}_\delta = \delta^\gamma{}_\delta + \frac{1}{2}\epsilon_{\alpha\beta}(E^{\alpha\beta})^\gamma{}_\delta$, a spinor transforms as,

$$\psi \longrightarrow (1 + \frac{1}{4}\epsilon_{\alpha\beta}\gamma^\alpha\gamma^\beta)\psi$$

In other words, the spinor representation of the Lorentz group is given by $E^{\alpha\beta} \longrightarrow \frac{1}{2}\gamma^\alpha\gamma^\beta$. This is what we need to find D_μ. We already derived the expression for the covariant derivative of a vector field, and spinors simply correspond to a different representation. So replacing $E^{\alpha\beta}$ with their spinor-space analogues, we get the **spin-covariant derivative**,

$$D_\mu\psi = (\partial_\mu + \frac{1}{4}\omega_{\alpha\beta\mu}\gamma^\alpha\gamma^\beta)\psi$$

$\omega_\mu \equiv \frac{1}{4}\omega_{\alpha\beta\mu}\gamma^\alpha\gamma^\beta$ is the **spin connection form** (it is actually a *matrix* of one-forms), or just 'spin connection'.

Therefore the free Dirac Lagrangian in a general spacetime is given by,

$$\begin{aligned}\mathcal{L}_{Dirac} &= i\overline{\psi}\gamma^\alpha D_\alpha\psi \\ &= i\overline{\psi}\gamma^\alpha V_\alpha{}^\mu(\partial_\mu + \frac{1}{4}\omega_{\beta\delta\mu}\gamma^\beta\gamma^\delta)\psi\end{aligned}$$

where the γ's are the usual flat-space Dirac matrices. This can also be written as,

$$\mathcal{L}_{Dirac} = i\overline{\psi}\gamma^\mu(\partial_\mu + \frac{1}{4}\omega_{\alpha\beta\mu}\gamma^\alpha\gamma^\beta)\psi$$

where the curved-space gamma matrices are given by $\gamma^\mu \equiv \gamma^\alpha V_\alpha{}^\mu$ and generate the Clifford algebra associated with the spacetime metric $g_{\mu\nu}$: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

For a more rigorous mathematical formulation of the above ideas in terms of associated vector bundles, see the book [16].

6 The Dilaton Mechanism

We remind the reader that we change our metric convention for this section.

In [2], methods are given for separately localising fermions and (Abelian) gauge bosons in the context of a Randall-Sundrum-like spacetime. The former are still localised by the familiar coupling to a kink, while a different scalar field, the dilaton, is required to localise the latter. If π represents the dilaton, localisation is achieved by multiplying the usual gauge-kinetic term by $e^{-K\pi}$, and finding a classical solution for π which allows for sensible dimensional reduction to be performed.

The first step towards building a realistic theory using this idea is to localise fermions and gauge bosons simultaneously, which was not done in [2]. This raises the question of whether the dilaton mechanism can be used to localise both types of particles. This would be desirable for future generalisation of the model to a fully interacting theory with non-Abelian gauge fields[16].

So we decided to start with the exact solution of the kink-gravity-dilaton system given in [2], and see if fermions could be localised by the dilaton in the same way that gauge bosons are. There is one minor technical point to be made here:

Usually, the kinetic term for a fermion can be taken to be $-a\overline{\Psi}\Gamma^A D_A\Psi + b(D_A\overline{\Psi})\Gamma^A\Psi$ for any a and b such that $a + b = 1$. This is because any two such Lagrangians differ by a multiple of $D_A(\overline{\Psi}\Gamma^A\Psi)$, and as such are equivalent. In our case though, we want to multiply the kinetic term by another field. The equations of motion will then depend on which values of a and b are chosen. The most natural choice is the 'symmetric' one with $a = b = \frac{1}{2}$, so this is what we used[17].

[16] After dimensional reduction, wavefunctions and coupling constants must be (finitely) renormalised to obtain the correctly normalised four-dimensional action. If fermions and gauge bosons are localised by different mechanisms, we are likely to end up with a different effective gauge coupling constant for gauge boson self-interactions than for gauge boson-fermion interactions, and this is ruled out by experiment.

[17] Notice that if we make the usual choice $a = 1, b = 0$, the equation of motion for Ψ is not affected by the presence of the dilaton.

6.1 A model with dilaton-fermion coupling

We are using the kink-gravity-dilaton solution of Kehagias & Tamvakis [2] as a classical background for our model. Thus we have the metric,

$$G_{MN} = \begin{pmatrix} e^{2A(y)}\eta_{\mu\nu} & 0 \\ 0 & e^{2B(y)} \end{pmatrix}$$

and scalar fields η, π (η is the field which forms the kink, and π is the dilaton), where,

$$\begin{aligned} \eta(y) &= v\tanh(ay) \\ A(y) = 4B(y) &= -\beta\left[\log(\cosh^2(ay)) + \tfrac{1}{2}\tanh^2(ay)\right] \\ \pi(y) &= \sqrt{3M^3}\beta\left[\log(\cosh^2(ay)) + \tfrac{1}{2}\tanh^2(ay)\right] \qquad \beta \equiv \tfrac{v^2}{36M^3} \end{aligned}$$

M is the fundamental mass scale of the theory, and a, v are positive parameters. We want to localise fermions via the same dilaton mechanism that localises gauge bosons, so we employ the same coupling of π to the kinetic term, to get the fermion action,

$$S_f = \int d^4x \int dy \sqrt{-G}\frac{1}{2}e^{-K\pi}\left\{-\overline{\Psi}\Gamma^A D_A\Psi + (D_A\overline{\Psi})\Gamma^A\Psi\right\}$$

where $K = \frac{\lambda}{2\sqrt{3M^3}}$, and λ is a dimensionless dilaton coupling constant. We have $\Gamma^\alpha = \gamma^\alpha$, $\Gamma^4 = \gamma^5$ (the γ are the usual 4D Dirac matrices) and $D_A = V_A{}^M(\partial_M + \omega_M)$, with the vielbein $V_A{}^M$ and spin connection ω_M given by,

$$\begin{aligned} V_A{}^\mu &= \delta^\mu_A e^{-A(y)} \\ V_A{}^4 &= \delta^4_A e^{-B(y)} \\ \omega_\mu &= \frac{A'(y)}{2}e^{A(y)-B(y)}\delta^\alpha_\mu\gamma_\alpha\gamma^5 \\ \omega_4 &= 0 \end{aligned}$$

If we vary $\overline{\Psi}$, we get,

$$\begin{aligned} \delta S_f &= \int d^4x \int dy \sqrt{-G}\frac{1}{2}e^{-K\pi}\left\{-\delta\overline{\Psi}\Gamma^A D_A\Psi + D_A(\delta\overline{\Psi})\Gamma^A\Psi\right\} \\ &= \int d^4x \int dy \sqrt{-G}e^{-K\pi}\delta\overline{\Psi}\left\{-\Gamma^A D_A\Psi + \frac{K}{2}D_A\pi\Gamma^A\Psi\right\} \end{aligned}$$

where we have discarded the surface term that comes from $D_A\left(\frac{1}{2}e^{-K\pi}\delta\overline{\Psi}\Gamma^A\Psi\right)$. Requiring $\delta S_f = 0$ then gives us the equation of motion for Ψ:

$$\Gamma^A D_A\Psi - \frac{K}{2}D_A\pi\Gamma^A\Psi = 0$$

Note that because π is a scalar, we simply have $D_A\pi = V_A{}^M\partial_M\pi$. To search for a localised zero-mode, let $\Psi(x,y) = f(y)\psi(x)$ where f is a scalar and ψ is a massless $4D$ fermion, so that $\gamma^\alpha\partial_\alpha\psi = 0$. Then the equation reduces to,

$$\left(f' + 2A'f - \frac{K\pi'}{2}f\right)\gamma^5\psi = 0$$

where primes denote differentiation with respect to y. So regardless of the chirality of ψ, we require,

$$f' + \left(2A' - \frac{K\pi'}{2}\right)f = 0$$

This equation is easily solved:

$$f = Ce^{-2A+\frac{K\pi}{2}}$$

for some constant C. We have $K\pi = -\frac{\lambda}{2}A$, so,

$$f = Ce^{(-2-\frac{\lambda}{4})A}$$

6.1.1 Dimensional reduction

We want to know if we get an effective four-dimensional action for ψ from this solution. To do this we need to integrate over y. Notice that all functions of y we have (A, B, f, π) are even, and that this means their first derivatives are odd. If we disregard the integral of odd functions over the whole real line, then our action reduces to,

$$S_f = \int d^4x \int dy \frac{1}{2} e^{3A+B} e^{-K\pi} f^2 \left\{ -\overline{\psi}\gamma^\alpha \partial_\alpha \psi + (\partial_\alpha \overline{\psi})\gamma^\alpha \psi \right\}$$

where we have substituted $\sqrt{-G} = e^{4A+B}$, and the extra factor of e^{-A} has come from the vielbein i.e. $\Gamma^\alpha V_\alpha{}^\mu = e^{-A}\delta^\mu_\alpha \gamma^\alpha$. So to get a finite four-dimensional action, we require that,

$$\int dy e^{3A+B-K\pi} f^2 < \infty$$

Substituting $B = \frac{A}{4}$, $-K\pi = \frac{\lambda}{2}A$ as well as our solution for f, we get,

$$e^{3A+B-K\pi} f^2 = C^2 e^{(3+\frac{1}{4}+\frac{\lambda}{2})A} e^{(-4-\frac{\lambda}{2})A} = C^2 e^{-\frac{3A}{4}}$$

Notice that this is independent of λ; we see that the dilaton fails to have any effect on the localisability of fermions. Indeed, the form of A indicates that the above expression diverges as $|y| \to \infty$, so we do not obtain an effective four-dimensional action.

6.2 Fermion localisation by a kink in a dilaton background

Because the above mechanism fails to localise fermions, we are forced to consider a model where fermions and gauge bosons are coupled to two different scalar fields – a kink and a dilaton respectively – in the hope to localise both simultaneously. The presence of the dilaton affects the gravitational background, so this raises the question of whether fermion localisation persists in this case.

We assume the kink-gravity-dilaton background as above, and take the action for our fermions to be,

$$S_f = \int d^4x \int dy \sqrt{-G} \left\{ -\overline{\Psi}\Gamma^A D_A \Psi - h\eta \overline{\Psi}\Psi \right\}$$

The resulting Dirac equation is,

$$\Gamma^A D_A \Psi + h\eta \overline{\Psi}\Psi = 0$$

As before, we investigate the behaviour of the zero-mode by writing $\Psi(x, y) = f(y)\psi(x)$, and demanding that $\gamma^\alpha \partial_\alpha \psi = 0$. The resulting equation for f depends on the chirality of ψ (remembering that $\Gamma^4 = \gamma^5$):

$$f' + (2A' \pm h\eta e^B) f = 0 \quad \text{for } \gamma^5\psi = \pm\psi$$

The solutions are,

$$f(y) = C \exp\left(-2A(y) \mp h \int_{y_0}^{y} dy' \eta(y') e^{B(y')} \right)$$

where C is a constant. The integral in the exponent is quite complicated, but we only need the asymptotic form of f anyway, so there is no need to evaluate it explicitly. Inspecting the solutions for η and B, we can see that as $|y| \to \infty$, η simply asymptotes to a constant, while $e^B \sim \cosh(ay)^{-2\beta}$. In turn, $\cosh(ay)^{-2\beta} \sim e^{-2a\beta|y|} \to 0$. So $\int_{y_0}^{y} dy' \eta(y') e^{B(y')} \to$ constant, and thus, independent of the chirality of ψ, we get that $f \sim e^{-2A}$ as $|y| \to \infty$. In the same way as before, the condition for the existence of the four-dimensional fermion is,

$$\int dy e^{3A+B} f^2 < \infty$$

But $B = \frac{A}{4}$, so the behaviour of the integrand as $|y| \to \infty$ is,

$$\begin{aligned} e^{3A+B} f^2 &\sim e^{3A+\frac{A}{4}} e^{-4A} \\ &= e^{-\frac{3A}{4}} \end{aligned}$$

This clearly diverges, so the integral is infinite. Therefore, in the presence of the dilaton, the kink fails to localise fermions. If a realistic theory is to be obtained, a different method must be utilised to localise the gauge bosons, the fermions, or both.

7 The Dvali-Shifman Mechanism

In section 4.5, we presented a different mechanism for localisation of gauge bosons, originally proposed by Dvali and Shifman [1]. Their model has $U(1)$ gauge bosons localised inside a bulk $SU(2)$ theory, but the idea generalises immediately to any gauge group G and appropriate subgroup H. One approach that seems quite natural is to consider a grand unified theory in the bulk, with the symmetry broken to the standard model gauge group on the wall.

In this section we find an explicit solution to the original $SU(2)$ model, before presenting a more realistic model of a bulk $SU(5)$ theory in which standard model gauge bosons are localised on the wall.

7.1 Breaking $SU(2)$ to $U(1)$

The argument in [1] (reproduced in section 4.5) seems sound, but as far as we can tell nobody has presented a solution with the required properties in the literature. We decided this would be a good first step.

The potential used previously isn't the most general quartic $SU(2) \times \mathbb{Z}_2$-invariant potential for this system. For completeness, we *did* work with the most general such potential, which can be written[18],

$$V(\eta, \chi) = (c\eta^2 - \mu^2)Tr(\chi^2) + \lambda[Tr(\chi^2)]^2 + \lambda'(\eta^2 - v^2)^2 \tag{4}$$

where all parameters are positive. Note that the constants here are not the same as those in the original Dvali-Shifman Lagrangian. In fact, they don't even have the same dimensions, because we are now working in five spacetime dimensions.

We want to find a stable static solution that depends only on y. Stability requires that our boundary conditions ($\chi = 0, \eta = \pm v$ as $y \to \pm\infty$) correspond to the global minimum of the potential. The following conditions are sufficient to guarantee this, and we will assume they hold from now on:

$$cv^2 - \mu^2 > 0 \qquad 4\lambda'\lambda - c^2 > 0$$

We can simplify the mathematics using gauge freedom. The adjoint rep. of $SU(n)$ consists of $n \times n$ traceless Hermitian matrices τ, acted on by $U \in SU(n)$ via $\tau \longrightarrow U\tau U^\dagger$. We choose a basis $\{\tau_j\}$ for $n = 2$ such that $Tr(\tau_j \tau_k) = \frac{1}{2}\delta_{jk}$[19], and write $\chi = \sum_{j=1}^{3} \chi_j \tau_j$. It is clear that we can perform a gauge transformation to diagonalise χ at each point. Thanks to the traceless condition, this leaves just a single degree of freedom for χ, usually labelled χ_3. When solving the equations, we will assume that this is the only non-zero element of χ. This line of reasoning explains why only perturbations of χ_3 were considered in [1]. The equations that follow from the above are then,

$$\begin{aligned} \frac{d^2\eta}{dy^2} &= \eta\left[4\lambda'(\eta^2 - v^2) + c\chi_3^2\right] \\ \frac{d^2\chi_3}{dy^2} &= \chi_3\left[\lambda\chi_3^2 + c\eta^2 - \mu^2\right] \end{aligned}$$

and we impose the boundary conditions $\chi_3 \to 0, \eta \to \pm v$ as $y \to \pm\infty$.

7.1.1 Numerical solutions and stability

Unfortunately, the equations above proved to be too hard to solve analytically, so we had to resort to numerical techniques. In this way, we obtained solutions which have the correct qualitative features. The program used was written in C, and implements the relaxation method. A brief overview is given in appendix C.

For the sake of obtaining numerical solutions, it is best to non-dimensionalise the equations. This was done by measuring all quantities in units of μ raised to the appropriate power. This is superficially equivalent to setting $\mu = 1$ in the above equations. Having done this, a solution was found using the parameter values,

$$\lambda = \frac{.7}{\mu} \qquad \lambda' = \frac{.15}{\mu}$$
$$c = \frac{.6}{\mu} \qquad v = 1.7\mu^{\frac{3}{2}}$$

The solutions for η, χ_3 are shown in Figure 1. Of course, we require the solution to be stable, in

[18] A term proportional to $Tr(\chi^4)$ is 'missing', but for an $SU(2)$ adjoint, this is proportional to $[Tr(\chi^2)]^2$

[19] These are usually taken to be the Pauli spin matrices, suitably normalised.

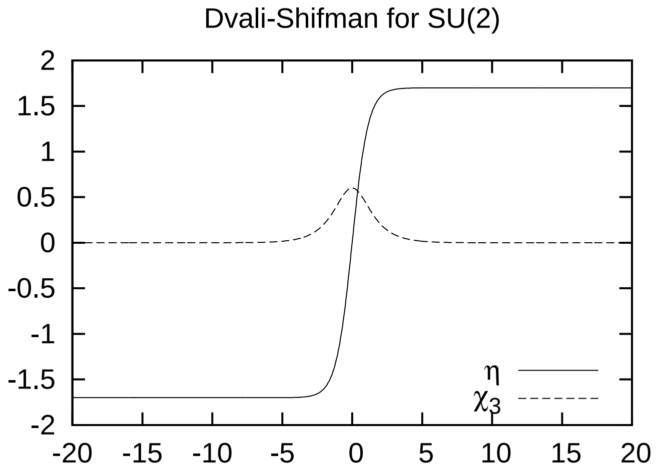

Figure 1: A solution which breaks $SU(2)$to $U(1)$ on a domain wall.

order to later quantise the theory. We will demonstrate stability by considering normal modes of time-dependent perturbations, and showing that these are all oscillatory. Normal modes could be collective oscillations of the fields, so perturbing the fields individually will not be sufficient, and we have to proceed as follows.

Define $\Theta = (\eta \quad \chi_3)^T$, and let $\Theta^0 = (\eta^0 \quad \chi_3^0)^T$ denote the static solution we have found. Now introduce a perturbation (we stick to only y-dependence),

$$\Theta = \Theta^0 + \epsilon e^{i\omega t} P(y)$$

where $P = (P_1 \quad P_2)^T$ gives the direction of the perturbation in field space. The equations of motion are,

$$-\frac{\partial^2 \Theta_j}{\partial y^2} + \frac{\partial V}{\partial \Theta_j} = -\frac{\partial^2 \Theta_j}{\partial t^2}$$

which becomes,

$$\frac{d^2 \Theta_j^0}{dy^2} + \epsilon e^{i\omega t} \frac{d^2 P_j}{dy^2} + \frac{\partial V}{\partial \Theta_j} = \epsilon \omega^2 e^{i\omega t} P_j$$

We can write $\frac{\partial V}{\partial \Theta_j} = \frac{\partial V}{\partial \Theta_j^0} + \epsilon e^{i\omega t} f(P) + O(\epsilon^2)$. The above equation is then satisfied identically to zeroth order in ϵ. We want to solve it to first order. f is linear in the perturbations, so there exists a matrix M such that $f(P) = MP$. We find that P satisfies the equation,

$$\left(-\frac{d^2}{dy^2} + M \right) P = \omega^2 P$$

We treat this as an eigenvalue problem, and find the possible eigenvalues ω^2. If any of these are negative, the solution is unstable, for the reasons given in section 4.5. Notice that the differential operator is Hermitian, because M is necessarily real and symmetric, so we are guaranteed to get real eignenvalues.

M is easily found by writing out $\frac{\partial V}{\partial \Theta_j}$ to first order in ϵ. For example,

$$\begin{aligned} \frac{\partial V}{\partial \Theta_0} &= \frac{\partial V}{\partial \eta} \\ &= \eta^0 \left[4\lambda'((\eta^0)^2 - v^2) + c(\chi_3^0)^2 \right] \\ &\quad + \epsilon e^{i\omega t} \left[\left(12\lambda'(\eta^0)^2 + c(\chi_3^0)^2 - 4\lambda' v^2 \right) P_0 + 2c\chi_3^0 \eta^0 P_1 \right] + O(\epsilon^2) \end{aligned}$$

In this way, we find that,

$$M = \begin{pmatrix} 12\lambda'(\eta^0)^2 + c(\chi_3^0)^2 - 4lv^2 & 2c\chi_3^0\eta^0 \\ 2c\chi_3^0\eta^0 & 3\lambda(\chi_3^0)^2 + c(\eta^0)^2 - \mu^2 \end{pmatrix}$$

Solving the eigenvalue equation numerically, the lowest eigenvalues we find are (to four decimal places),

$$\begin{aligned}\omega^2 = 0.0000, &\quad 0.2353, \quad 0.7375\\ 0.7382, &\quad 0.7483, \quad 0.7508\\ 0.7664, &\quad 0.7718, \quad 0.7923\end{aligned}$$

The are two features of this partial spectrum that deserve comment. The first is the zero eigenvalue. This is expected, and is harmless. It corresponds to the 'zero mode' of the solution. The equations of motion of the system are translationally invariant in the extra dimension, but the solution we have found is not. This means there is a whole family of degenerate static solutions that differ only by translation. Therefore translating our solution in the extra dimension would cost no energy, and this is the origin of the zero eigenvalue. For more details, see the discussion of the zero mode of the $(1+1)$-dimensional kink in [5].

The second feature worth mentioning is the fact that except for the first few, the differences between the increasing eigenvalues are quite small. This continues for all the larger eigenvalues. We are seeing the numerical equivalent of the continuum of eigenvalues expected for a 'finite well' Schrödinger equation, which is effectively what we have solved. We will show this more clearly when we examine the $SU(5)$ case.

The lack of any negative eigenvalues indicates that the solution we have found is stable, thus verifying that the Dvali-Shifman mechanism indeed works as claimed.

7.2 Breaking $SU(5)$ to $SU(3) \times SU(2) \times U(1)$

Having proved that the Dvali-Shifman idea works as expected in the $SU(2)$ case, we proceed to a potentially realistic model, where the unbroken gauge group is that of the standard model. For this we will require a larger group in the bulk, and the obvious choice is the grand unification group $SU(5)$, described in section 3.

First we briefly re-iterate what we are trying to achieve. We will look for solutions such that the symmetry is broken to $SU(3)_c \times SU(2)_L \times U(1)_Y$ on a domain wall formed by a gauge singlet scalar field. This symmetry is exact everywhere, so the standard model gauge fields will remain massless. $SU(5)$ is unbroken in the bulk, so the theory away from the wall should exhibit confinement, with all physical states having masses of order the confinement scale Λ. Therefore at low energies, the massless standard model fields cannot propagate in the bulk, and we get an effective (3+1)-dimensional theory on the wall. We now proceed to find such a solution.

Again, we introduce two scalar fields: χ in the adjoint representation, and η which is a gauge singlet. As well as gauge invariance, we also impose the $\mathbb{Z}_2$ symmetry $\eta \longrightarrow -\eta$ again, which will lead to kink-like solutions for η.

Before we can write down the potential of the theory, we note that there are two extra terms compared to the $SU(2)$ case – terms proportional to $Tr(\chi^3)$ and $Tr(\chi^4)$. For $SU(2)$, the first of these is identically zero, and the second is proportional to $[Tr(\chi^2)]^2$, and so not independent. For simplicity we impose an additional $\mathbb{Z}_2$ symmetry $\chi \longrightarrow -\chi$, which disallows the $Tr(\chi^3)$ term. The potential of the theory is then given by,

$$V(\eta, \chi) = (c\eta^2 - \mu^2)Tr(\chi^2) + \lambda_1 \left[Tr(\chi^2)\right]^2 + \lambda_2 Tr(\chi^4) + \lambda'(\eta^2 - v^2)^2$$

We assume that all parameters are positive; this trivially guarantees that V is bounded from below. Again, we want $\eta^2 = v^2$ and $\chi = 0$ to correspond to the global minimum of the potential. Additional constraints which ensure this are,

$$cv^2 - \mu^2 > 0 \qquad 4\lambda'\lambda_1 - c^2 > 0$$

As before, we can use gauge freedom to put χ into a standard form. The $SU(5)$ adjoint consists of traceless Hermitian matrices, and transforms via $\chi \longrightarrow U\chi U^\dagger$ so χ can be diagonalised by a gauge transformation. Explicitly, we write $\chi = \sum_{j=1}^{4} \chi_j \tau_j$, where,

$$\tau_1 = \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \quad \tau_2 = \frac{1}{2} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

$$\tau_3 = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \quad \tau_4 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

The τ's have been chosen so that $Tr(\tau_j \tau_k) = \frac{1}{2}\delta_{jk}$. In terms of these four components, the potential is (brace yourself, dear reader):

$$\begin{aligned} V(\eta, \chi) &= \frac{1}{2}(c\eta^2 - \mu^2)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2) \\ &+\lambda_1 \left[\frac{1}{4}(\chi_1^4 + \chi_2^4 + \chi_3^4 + \chi_4^4) + \frac{1}{2}(\chi_1^2\chi_2^2 + \chi_1^2\chi_3^2 + \chi_1^2\chi_4^2 + \chi_2^2\chi_3^2 + \chi_2^2\chi_4^2 + \chi_3^2\chi_4^2)\right] \\ &+\lambda_2 \left[\frac{7}{120}\chi_1^4 + \frac{9}{20}\chi_1^2\chi_2^2 + \frac{1}{8}\chi_2^4 + \frac{1}{5}\chi_1^2\chi_3^2 + \frac{1}{8}\chi_3^4 + \frac{1}{\sqrt{5}}\chi_1\chi_3^2\chi_4 + \frac{1}{5}\chi_1^2\chi_4^2 + \frac{1}{4}\chi_3^2\chi_4^2 \right. \\ &\left. - \frac{1}{3\sqrt{5}}\chi_1\chi_4^3 + \frac{1}{8}\chi_4^4\right] + \lambda'(\eta^2 - v^2)^2 \end{aligned}$$

Recall that the Klein-Gordon equation in the presence of a potential is $\partial_M \partial^M \phi + \frac{\partial V}{\partial \phi} = 0$. We are looking for static solutions that depend only on the extra dimensional coordinate y, and in this case the equations following from the above potential are,

$$\begin{aligned} \frac{d^2\eta}{dy^2} &= \eta \left\{4\lambda'(\eta^2 - v^2) + c\sum_{j=1}^{4} \chi_j^2\right\} \\ \frac{d^2\chi_1}{dy^2} &= \chi_1 \left\{\lambda_1 \sum_{j=1}^{4} \chi_j^2 + \lambda_2 \left(\frac{7}{30}\chi_1^2 + \frac{9}{10}\chi_2^2 + \frac{2}{5}\chi_3^2 + \frac{2}{5}\chi_4^2\right) + c\eta^2 - \mu^2\right\} \\ &+ \frac{1}{\sqrt{5}}\lambda_2\chi_4 \left(\chi_3^2 - \frac{1}{3}\chi_4^2\right) \\ \frac{d^2\chi_2}{dy^2} &= \chi_2 \left\{\lambda_1 \sum_{j=1}^{4} \chi_j^2 + \lambda_2 \left(\frac{9}{10}\chi_1^2 + \frac{1}{2}\chi_2^2\right) + c\eta^2 - \mu^2\right\} \\ \frac{d^2\chi_3}{dy^2} &= \chi_3 \left\{\lambda_1 \sum_{j=1}^{4} \chi_j^2 + \lambda_2 \left(\frac{2}{5}\chi_1^2 + \frac{1}{2}\chi_3^2 + \frac{2}{\sqrt{5}}\chi_1\chi_4 + \frac{1}{2}\chi_4^2\right) + c\eta^2 - \mu^2\right\} \\ \frac{d^2\chi_4}{dy^2} &= \chi_4 \left\{\lambda_1 \sum_{j=1} 4\chi_j^2 + \lambda_2 \left(\frac{2}{5}\chi_1^2 + \frac{1}{2}\chi_3^2 - \frac{1}{\sqrt{5}}\chi_1\chi_4 + \frac{1}{2}\chi_4^2\right) + c\eta^2 - \mu^2\right\} \\ &+ \frac{1}{\sqrt{5}}\lambda_2\chi_1\chi_3^2 \end{aligned}$$

From the form of the τ's, we see that we will get breaking to the standard model gauge group if χ_1 has a non-zero value inside the wall, but the other χ's all remain zero. Therefore we will look for solutions of this form. The resulting equations are,

$$\begin{aligned} \frac{d^2\eta}{dy^2} &= \eta \left\{4\lambda'\eta^2 + c\chi_1^2 - 4\lambda' v^2\right\} \\ \frac{d^2\chi_1}{dy^2} &= \chi_1 \left\{\left(\lambda_1 + \frac{7}{30}\lambda_2\right)\chi_1^2 + c\eta^2 - \mu^2\right\} \end{aligned} \tag{5}$$

7.2.1 Numerical solutions and stability

Notice that the equations are very nearly identical to the equations we got in the $SU(2)$ theory. The stability issues will be more complicated, because we now have three extra fields, but it turns out

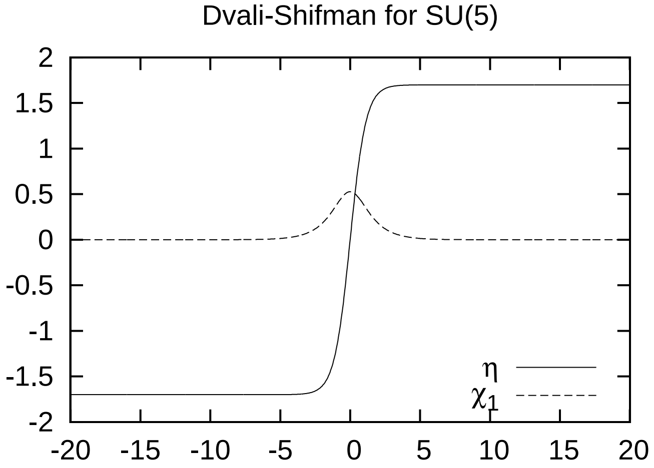

Figure 2: A solution for η and χ which breaks $SU(5)$ to the standard model gauge group on a domain wall.

that the following choice of parameter values gives us what we require:

$$\lambda_1 = \frac{.7}{\mu} \qquad \lambda_2 = \frac{.7}{\mu}$$
$$\lambda' = \frac{.15}{\mu} \qquad c = \frac{.6}{\mu}$$
$$v = 1.7\mu^{\frac{3}{2}}$$

The corresponding numerical solution is plotted in Figure 2.

Now that we have a static solution of the necessary form, we need to demonstrate that it is stable. This proceeds similarly to the $SU(2)$ case, except that we now have five fields instead of just two. Our task is simplified by the fact that our solution has $\chi_2 = \chi_3 = \chi_4 \equiv 0$. This means that perturbatively, these fields don't couple to each other, η, or χ_1.

In detail, when we let $\Theta = (\eta, \chi_1, \chi_2, \chi_3, \chi_4)^T$ and again write down our perturbation eigenvalue equation,

$$\left(-\frac{d^2}{dy^2} + M\right) P = \omega^2 P$$

we find that M is in block-diagonal form; it contains a 2×2 block in the top left corner, corresponding to coupled oscillations of η and χ_1, and a *diagonal* 3×3 block in the bottom right, corresponding to independent oscillations of the other fields. In this way, our eigenvalue equation actually separates into the four independent equations,

$$\begin{aligned}
\left[-\frac{d^2}{dy^2} + \tilde{M}\right]\begin{pmatrix} P_0 \\ P_1 \end{pmatrix} &= \omega_1^2 \begin{pmatrix} P_0 \\ P_1 \end{pmatrix} \\
\left[-\frac{d^2}{dy^2} + \left(\lambda_1 + \frac{7}{30}\lambda_2\right)(\chi_1^0)^2 + c(\eta^0)^2 - \mu^2\right] P_2 &= \omega_2^2 P_2 \\
\left[-\frac{d^2}{dy^2} + \left(\lambda_1 + \frac{2}{5}\lambda_2\right)(\chi_1^0)^2 + c(\eta^0)^2 - \mu^2\right] P_3 &= \omega_3^2 P_3 \\
\left[-\frac{d^2}{dy^2} + \left(\lambda_1 + \frac{2}{5}\lambda_2\right)(\chi_1^0)^2 + c(\eta^0)^2 - \mu^2\right] P_4 &= \omega_4^2 P_4
\end{aligned}$$

where

$$\tilde{M} = \begin{pmatrix} 12\lambda'(\eta^0)^2 - 4\lambda' v^2 + c(\chi_1^0)^2 & 2c\chi_1^0\eta^0 \\ 2c\chi_1^0\eta^0 & \left(3\lambda_1 + \frac{7}{10}\lambda_2\right)(\chi_1^0)^2 + c(\eta^0)^2 - \mu^2 \end{pmatrix}$$

An eigenvalue of any of these equations is also an eigenvalue of the original equation, so stability of our solution requires that all eigenvalues of these four equations are positive. These eigenvalues can be found numerically, and the lowest for each equation are displayed in Table 1. We see the

ω_1^2	ω_2^2	ω_3^2	ω_4^2
0.0000	0.0856	0.0216	0.0216
0.2396	0.7368	0.7362	0.7362
0.7376	0.7382	0.7381	0.7381
0.7382	0.7460	0.7449	0.7449
0.7483	0.7506	0.7506	0.7506
0.7508	0.7628	0.7614	0.7614
0.7665	0.7714	0.7712	0.7712
0.7718	0.7877	0.7862	0.7862
0.7923	0.8004	0.8002	0.8002

Table 1: Eigenvalues for perturbations of the $SU(5)$ solution.

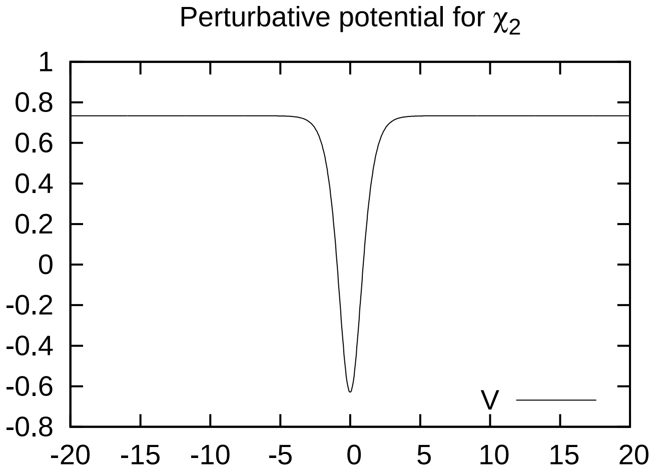

Figure 3: The effective potential for perturbations of χ_2 in the static background solution found for η and χ_1.

same structure as in the $SU(2)$ case. There is a zero mode, corresponding to translation of the solution for η and χ_1 (the reason such a zero mode does not occur for the other fields is that their solutions – identically zero – are translationally invariant), plus a handful of 'bound states', and then a continuum of eigenvalues.

To clarify these features, the effective potential for perturbations of χ_2 is plotted in Figure 3. The potential is just a smooth finite well, and looking at the eigenvalues we can see it must support a single bound state, which nonetheless (importantly!) has positive energy. Asymtotically, $V \simeq .73$, and this is approximately the value at which the 'continuum' eigenvalues start. These are familiar features of a one-dimensional quantum mechanical system. To confirm that we really are getting a continuum of eigenvalues, the code was run again, with the domain doubled. This should allow more continuum modes to 'fit', so the numerical eigenvalues should be closer together. The resulting lowest eigenvalues are,

$$\begin{array}{lll} \omega_2^2 = 0.0856 & 0.7348 & 0.7350 \\ 0.7373 & 0.7380 & 0.7416 \\ 0.7431 & 0.7478 & 0.7502 \end{array}$$

As expected, the bound state eigenvalue has not changed, but the higher eigenvalues are more tightly packed, giving a better numerical approximation to a continuum.

Adding fermions

At this stage, our model contains only scalar fields. We have argued that when gauge bosons are incorporated into the theory, they will be localised to the domain wall, yielding an effective $(3+1)$-dimensional theory. We have not commented on fermions though. This may seem folly, for we have seen in the case of the dilaton that results from a toy model won't necessarily hold when extra fields are added.

In the case of our $SU(5)$ model though, this shouldn't be a problem. There is no apparent reason that the Yukawa coupling to the kink field would fail to localise fermions, but even if it did, it shouldn't matter. The massless zero modes of the fermions should be localised to the domain wall for the same reason the gauge bosons are — to propagate in the bulk, they would have to be bound into 'colourless' states with masses of the order of the confinement scale of the theory. The same line of reasoning was used in [1] in the context of the $SU(2)$ model.

7.3 Miscellanea

We have achieved our initial goal with the $SU(5)$ model, but there is another possibility that deserves comment.

In the usual symmetry breaking scenario, the scalar field assumes a uniform value which minimises the potential. In this context, Li has proven that for a field in the adjoint representation, $SU(5)$ always breaks to $SU(3) \times SU(2) \times U(1)$ if $\lambda_2 > 0$ [17]. In our model though, χ varies in the extra dimension, so Li's results do not apply. This suggests the possibility that we could get *more* symmetry breaking than in the traditional model.

It may seem undesirable to break the standard model gauge group, but there is one interesting possibility. It is conceivable that we could use the Dvali-Shifman mechanism to break $SU(5)$ all the way to $SU(3) \times U(1)_Q$ — the unbroken gauge group we observe in nature. Unfortunately, inspection of the τ's reveals that the best we can potentially do with an adjoint Higgs is break to $SU(3) \times U(1) \times U(1)$, by finding a solution for which both χ_1 and χ_2 are non-zero. Still, if we could achieve this, it would be an interesting proof of concept.

Solutions were found in which χ_1 and χ_2 were both non-zero, but unfortunately the perturbative analysis showed them all to be unstable. It seems probable that no stable solutions of this form exist, but this could only be verified by a systematic scan of parameter space, or analytical results.

If viable, the additional symmetry breaking may actually be practical in a different model. The initial approach taken in this project was to consider an $SO(10)$ model in the bulk. In an $SO(10)$ model, Li's results imply that 'normal' symmetry breaking to $SU(3) \times SU(2) \times U(1)$ must occur in two stages, involving two different scalar field multiplets. We thought the Dvali-Shifman mechanism might provide a way to break $SO(10)$ directly to the standard model. The initial results didn't seem promising, so the focus was shifted to $SU(5)$.

8 What About Gravity?

We have managed to obtain classical solutions of the $SU(5)$ theory which, via the Dvali-Shifman mechanism, should lead to localised fermions and gauge bosons in the quantised theory. But we have done this in a flat spacetime. If our model is to be realistic, we have to include the Randall-Sundrum idea so that gravity is also localised. In this section we set up such a model. The resulting equations have not yet been solved.

8.1 $SU(5)$ Dvali-Shifman in a warped spacetime

The existence of the Randall-Sundrum solution depends crucially on having a non-zero value of the five-dimensional cosmological constant, so we expect that we will require this also. We take the action for the theory to be[20],

$$S = \int d^4x \int dy \sqrt{G} \left\{ -\Lambda - 2M^3 R + \frac{1}{2} G^{MN} D_M \eta D_N \eta + G^{MN} Tr\left[D_M \chi D_N \chi \right] - V(\eta, \chi) \right\}$$

Here G is the determinant of the metric, M is the fundamental mass scale of the theory, and Λ is the five-dimensional cosmological constant. Varying the action with respect to G^{MN} gives Einstein's equations for the metric,

$$\begin{aligned} R_{MN} - \frac{1}{2} G_{MN} R &= \frac{1}{4M^3} G_{MN} \left\{ \Lambda - \frac{1}{2} G^{PQ} D_P \eta D_Q \eta - G^{PQ} Tr\left[D_P \chi D_Q \chi \right] + V(\eta, \chi) \right\} \\ &\quad + \frac{1}{2M^3} \left\{ \frac{1}{2} D_M \eta D_N \eta + Tr\left[D_M \chi D_N \chi \right] \right\} \end{aligned} \qquad (6)$$

[20] Implicit in the sign of the Einstein-Hilbert term is our convention for the sign of the Riemann tensor: $(D_\kappa D_\tau - D_\tau D_\kappa) X^\mu = R^\mu{}_{\nu\kappa\tau} X^\nu$.

We take an ansatz of the same form as the Randall-Sundrum metric,

$$ds^2 = G_{MN} dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

Assuming, as we did in the case without gravity, that the scalar fields depend only on y, we can substitute this ansatz into (6) to obtain,

$$\sigma'' = \frac{1}{6M^3} \left\{ \frac{1}{2} \left(\frac{d\eta}{dy} \right)^2 + Tr \left[\left(\frac{d\chi}{dy} \right)^2 \right] \right\} \tag{7}$$

$$(\sigma')^2 = \frac{1}{24M^3} \left\{ -\Lambda + \frac{1}{2} \left(\frac{d\eta}{dy} \right)^2 + Tr \left[\left(\frac{d\chi}{dy} \right)^2 \right] - V(\eta, \chi) \right\} \tag{8}$$

As in the Randall-Sundrum case, a negative cosmological constant is essential to obtaining the type of solution we want. To see this we note that as $y \to \pm\infty$, $\frac{d\eta}{dy}$, $\frac{d\chi}{dy}$, and $V(\eta, \chi)$ all approach zero, so without a cosmological constant, σ would asymptote to a constant. We could not then obtain an effective four-dimensional theory, because the integral over y would be infinite.

The equations for the scalar fields are,

$$\frac{d^2\eta}{dy^2} - 4\sigma' \frac{d\eta}{dy} = \frac{\partial V}{\partial \eta} \tag{9}$$

$$\frac{d^2\chi_j}{dy^2} - 4\sigma' \frac{d\chi_j}{dy} = \frac{\partial V}{\partial \chi_j} \tag{10}$$

We now have a fairly horrendous set of coupled non-linear equations to solve – much more complicated than those in the case without gravity. We can however make a small amount of progress quite easily.

The fine-tuning condition in the Randall-Sundrum model comes about from requiring consistency of equations (7) and (8) . We would like to find the analogue of this condition in our model. Something interesting happens when we attempt to do this. Differentiating (8), and substituting (9) and (10) into the resulting expression, we obtain (7)! This means that (7) is redundant. It seems that when the 'brane' is generated by dynamical fields, the fine-tuning condition is replaced by the equations satisfied by the fields themselves. This is a very nice result – fine tuning of parameters is undesirable in any theory. Not only that, but we now have only three coupled equations to solve.

We have not yet attempted to solve the set of equations (8), (9), and (10).

9 Conclusion

In theories with extra dimensions, the main challenge is to explain why we only observe three spatial dimensions, in a way that doesn't contradict other experimental facts. The most natural way of achieving this from a field-theoretical point of view seems to be to trap the standard model fields (and gravity!) a $(3+1)$-dimensional topological defect.

In this thesis we have concentrated on the case of one extra dimension, where our universe is identified with the centre of a domain wall formed by a scalar field. Fermions are easily localised to the wall, but gauge bosons pose a bigger problem. An initial approach involving a second scalar field, the dilaton, was shown to interfere with the fermion localisation, and was thus abandoned.

Focus then shifted to developing a grand unified theory in which only the standard model gauge bosons would be localised to the wall. We succeeded in finding a solution to an $SU(5)$ GUT which exhibited the symmetry-breaking required for this to occur, and argued that localised fermions can also be incorporated into the model quite easily. Finally, we indicated how the theory would need to be extended to include gravity, although little work has been done in this direction as yet.

Apart from including gravity, there are other issues which need to be addressed to make this model realistic. In particular, we have not as yet solved the problem of how to subsequently break the electroweak symmetry of the standard model on the brane. Presumably this will require yet another scalar field, analogous to the standard model Higgs.

A Lie Groups

The concept of symmetry plays a major role in physics, and especially in modern particle physics. In particular, all fundamental forces are assumed to arise from 'gauge symmetries'. These symmetries are described by the action of certain Lie groups on the vector space of field operators. As such, it is important for physicists to have an understanding of certain aspects of the theory of Lie groups.

A.1 Elementary group theory

A group is a set endowed with a binary operation that satisfies certain axioms. We will denote the operation by '$\cdot$' so that $(a, b) \to a \cdot b$. A set G with such an operation is said to be a **group** under that operation if for all $a, b, c \in G$ the following axioms hold:

- $a \cdot b \in G$
- $\exists\ e \in G$ *such that* $e \cdot a = a \cdot e = a$
- *For all* $a \in G\ \exists\ a^{-1} \in G$ *such that* $a \cdot a^{-1} = a^{-1} \cdot a = e$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

We call e the **identity** in G and a^{-1} the **inverse** of a (or "a inverse"). Note that one of the rules familiar from elementary multiplication is missing - commutativity: $a \cdot b = b \cdot a$. Commutativity only holds in some groups; those groups are referred to as **Abelian** groups.

Some familiar examples may be helpful: a vector space is a group under addition; the integers $\mathbb{Z}$ form a group under addition; the non-zero real numbers $\mathbb{R}^*$ form a group under multiplication. These are all Abelian groups. An example of a set with a binary operation that fails to form a group is the set of integers under multiplication; inverses do not exist in this case.

In practice, the operation in non-Abelian groups is usually denoted simply by juxtaposition of elements, so that $(a, b) \to ab$, and in Abelian groups by the 'plus' symbol, so that $(a, b) \to a + b$. In general cases where it is not specified what type of group we have, the non-abelian notation is used.

If G is a group, then $H \subset G$ is called a **subgroup** of G, sometimes denoted $H \leq G$, if it forms a group itself under the group operation of G. It is easily seen that $H \subset G$ is a subgroup if, given any $a, b \in H$ the following hold:

- $a^{-1} \in H$
- $ab \in H$

Note that these two conditions guarantee that $e \in H$, so all subgroups contain the identity. A simple example of a subgroup is the group of even integers inside the group of all integers: the sum of two even integers is even, and the negative (inverse) of an even integer is even.

Given a subgroup $H \leq G$, it is often useful to define the space of its cosets. A **coset** of H in G is a subset of G of the following form. Given $g \in G$, we define:

$$gH = \{gh : h \in H\}$$

Because H is a subgroup, $e \in H$, and therefore $g \in gH$. If $g \in H$ then $gH = H$. More generally, although any element of G defines a unique coset, each coset can be specified by multiple elements. In fact, $g_1 H = g_2 H$ if and only if $g_1 g_2^{-1} \in H$. Each element that defines a coset is called a **representative** of that coset. The cosets of H form a partition of G; each element of G belongs to exactly one coset of H. These are all elementary facts, and can be easily proven.

It seems natural now to try to define a group structure on the set of cosets of H via the rule $g_1 H \cdot g_2 H = (g_1 \cdot g_2)H$. We run into a problem here though: it could be true that $g_3 H = g_2 H$ but $(g_1 \cdot g_3)H \neq (g_1 \cdot g_2)H$. In other words, our definition may depend on which representative of the coset we choose, which would make it not well defined. It turns out that the above multiplication is only well-defined if an extra condition is met, which leads us to another definition:

A subgroup $H \leq G$ is called a **normal subgroup** if for all $h \in H$ and all $g \in G$, $g^{-1}hg \in H$. Equivalently, $g^{-1}Hg = H$. It is easily checked that if H is a normal subgroup of G, then its cosets form a group under the operation previously defined. We denote this group G/H and call it the **quotient group** of G by H. Verbally, G/H is often referred to as "G mod H".

Another common object in group theory is the product of two groups. Given groups G_1, G_2, we define their **direct product** (or just 'product') by:

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

with group operation:

$$(g_1, g_2) \cdot (g_1', g_2') = (g_1 \cdot g_1', g_2 \cdot g_2')$$

A.2 Lie groups

A **Lie group** G is a differentiable manifold that is also a group such that multiplication and inversion ($a \to a^{-1}$) are both differentiable mappings (multiplication as a mapping from $G \times G$ to G and inversion as a mapping from G to itself). This may be a somewhat meaningless definition for those with no knowledge of differential topology, but this project deals only with very concrete examples of Lie groups, to which we now turn.

First we note that invertible $n \times n$ matrices with entries in either $\mathbb{R}$ or $\mathbb{C}$ form a group under matrix multiplication. These groups are called the **general linear groups**, and are denoted by $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ respectively. We will denote the identity matrix in any of these groups simply by 1. All Lie groups we deal with will be subgroups of these groups.

In particular, the following Lie groups occur frequently in particle physics:

- $U(n)$ - the group of all unitary $n \times n$ matrices.
- $O(n)$ - the group of all (real) orthogonal $n \times n$ matrices.

The groups $SU(n)$ and $SO(n)$ are the subgroups of the above consisting of matrices with determinant 1. We note in passing that $U(1)$ is simply the set of complex numbers of modulus 1.

With the exception of $SO(1), O(1), SU(1)$ and $SO(2) \cong U(1)$, the above groups are all non-Abelian. $SO(1), O(1)$, and $SU(1)$ are zero-dimensional Lie groups, and thus rather trivial, so the only Abelian Lie group of interest to us is $U(1)$.

The groups $O(n)$ and $U(n)$ have nice geometric interpretations. If the matrices in these groups are considered as representing linear transformations on $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively, then they are exactly the transformations that preserve the usual inner product on these spaces. In other words, they represent transformations analogous to rotations and reflections in familiar three-dimensional space.

A.2.1 Lie algebras

Lie groups can be very complicated mathematical objects. Fortunately, to every Lie group is associated a Lie algebra – a simpler object which nonetheless captures many of the important properties of the group.

A **Lie algebra** V is a vector space equipped with an operation, usually denoted by $[u, v]$, called the **Lie bracket**, which satisfies the following for any vectors $u, v, w \in V$, and any scalar a (belonging to either $\mathbb{R}$ or $\mathbb{C}$ depending on whether the vector space is real or complex)

- $[u, v] \in V$
- $[u, v] = -[v, u]$
- $[au + v, w] = a[u, w] + [v, w]$
- $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$

The last axiom in the above is given a special name - the **Jacobi identity**.

Once we have a basis $\{v^1, \ldots, v^k\}$ for V, we can define the **structure constants** C^{abc} of the Lie algebra by:

$$[v^a, v^b] = iC^{abc}v^c$$

where repeated indices are summed over. The Jacobi identity can now be written as an identity for the structure constants:

$$C^{abd}C^{dce} + C^{cad}C^{dbe} + C^{bcd}C^{dae} = 0$$

There is more we can say about the structure constants. It is immediate from the anti-symmetry of the Lie bracket that C^{abc} is anti-symmetric in a and b, but it turns out we can choose a basis in which it is anti-symmetric in all three indices. We won't prove this here.

The Lie algebra of a Lie group

The definition of the Lie algebra associated with a Lie group requires notions from differential topology, so we will give a definition that is equivalent for our purposes, and easier to understand. Given some Lie group G, we define the Lie algebra of G, denoted $\mathbf{g}$, as follows[21]:

$$\mathbf{g} = \{A : exp(-iA) \in G\}$$

where $exp()$ is defined by its usual power series, which converges for all square matrices. For all the groups that will be of interest to us, it is a non-trivial fact that all their elements can be obtained

[21] In mathematics, this definition usually does not contain the factor of $-i$. Physicists include this so that the Lie algebras of the unitary groups consist of Hermitian matrices.

by exponentiation of some matrix, so the Lie algebra contains all the information of the Lie group. The Lie bracket operation for these Lie algebras is defined by,

$$[A, B] = AB - BA$$

where AB denotes the matrix product of A and B. It is often convenient to choose a basis $\{\tau^1, \ldots, \tau^k\}$ such that $Tr(\tau^a \tau^b) = \frac{1}{2}\delta_{ab}$. To prove that this is possible, notice that $< A, B >= Tr(AB)$ defines an inner product on a vector space of square matrices, so the above is just equivalent to finding an orthonormal basis, which can always be done.

Abelian groups are quite trivial in a lot of ways, and we are often interested in exactly how a group departs from being Abelian. To this end, the **commutator** of two group elements a, b is defined to be the element $aba^{-1}b^{-1}$. Note that this is the identity element if and only if $ab = ba$ i.e. if a and b commute. So the commutators of elements measure how non-Abelian a group is. Let's study the commutator in Lie groups:

Let ϵ be an infinitesimal positive number, and take $A, B \in \mathbf{g}$. Then we can define two group elements "infinitesimally close to the identity" by:

$$U = exp(-i\epsilon A) = I - i\epsilon A + O(\epsilon^2)$$
$$V = exp(-i\epsilon B) = I - i\epsilon B + O(\epsilon^2)$$

We can now calculate the commutator of U and V. We find:

$$UVU^{-1}V^{-1} = 1 - \epsilon^2[A, B] + O(\epsilon^3)$$

This result suggests something which is in fact true, but requires more proof: two group elements $exp(-iA)$ and $exp(-iB)$ commute if and only if $[A, B] = 0$. Therefore the non-Abelian nature of a Lie group is captured by the Lie bracket of its Lie algebra.

A.3 Representations

We now need to look at what it means for a Lie group to act on a vector space. This leads immediately to the idea of a representation of a group.

Suppose some group G acts linearly on a vector space. This means that for any group element g and any vector v, we can obtain another vector $g \cdot v$ in a linear fashion. Therefore the action of G amounts to the action of a collection of linear transformations, which can be represented by square matrices. The action is related to the group multiplication by the rule $g \cdot (h \cdot v) = (gh) \cdot v$. This leads to the following definition:

A **linear representation** (or just representation) of a group is a function $\mathcal{D}$ from the group to some set of $n \times n$ matrices such that $\mathcal{D}(ab) = \mathcal{D}(a)\mathcal{D}(b)$. Therefore the matrices $\mathcal{D}(a)$ have the same multiplication properties as the group elements a. So the representation is literally a way of representing the group multiplication by multiplication of matrices. The group can now act on an n-dimensional vector space via the representation. Note though that the function $\mathcal{D}$ may not be one to one - for example, the **trivial representation** simply maps every element of G to the identity matrix.

Suppose then that we have a group G acting on a vector space V. It may be that V has some subspace $V' \subset V$ such that $G \cdot V' \subset V'$. In this case the representation is said to be **reducible**. Otherwise it is said to be **irreducible**. For the groups we are interested in, all representations are completely reducible, which means we can always write $V = V^1 \oplus V^2 \oplus \ldots \oplus V^l$ such that the representation on each V^i is irreducible.

For many of the groups relevant to particle physics, there is at most one irreducible representation of any given dimensionality. Therefore representations are often denoted by $\mathbf{n}$, where n is the dimension of the representation.

It is often very important to know how a representation breaks down into irreducible representations. Each irreducible component of a representation is essentially independent, so must be considered separately.

A.3.1 Important cases

There are a few concepts concerning representations that arise again and again in particle physics.

Fundamental representations

Each Lie group we consider is a subgroup of some general linear group $GL(n, \mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. As such, there is a canonical representation arising from the natural action of the group on n-dimensional vectors. This is called the **fundamental representation**.

Induced representations
Given a Lie group G and some subgroup H, any representation of G induces a representation of H in the obvious way. This is called an **induced representation**. It is important to note that even if the representation of G is irreducible, the induced representation of H will generally break down into several irreducible components.

Conjugate representations
Suppose $\mathcal{D}$ is a representation of G on a complex vector space V. Then we can define a representation $\mathcal{D}^*$ on the complex conjugate space V^* by $\mathcal{D}^*(a) \cdot v^* = (\mathcal{D}(a) \cdot v)^*$. This is called the **conjugate representation**.

Extending our notation from above, the conjugate representation of $\mathbf{n}$ is denoted by $\mathbf{n}^*$.

B Topological Defects

We mentioned earlier that the domain wall solution we have been using is an example of a 'topological defect'. In this appendix we explain the meaning of this term, and why such a solution is guaranteed to be stable. First we revisit the kink, and then present the general theory.

B.1 The kink

Suppose we have a field theory in $1+1$ dimensions, which contains just a single scalar field η. If we impose a $\mathbb{Z}_2$ symmetry $\eta \longrightarrow -\eta$, the Lagrangian of the theory is,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \lambda(\eta^2 - v^2)^2$$

where $\mu = 0, 1$. This model exhibits the phenomenon of spontaneous symmetry breaking; the vacuum manifold consists of the two points $\eta = \pm v$. We consider the problem of finding finite-energy static solutions of this (classical) theory. The equation satisfied by a static solution is,

$$\frac{d^2\eta}{dy^2} - 4\lambda\eta(\eta^2 - v^2) = 0$$

To have finite energy, the field must asymptote to one of its vacua as $y \to \pm\infty$. We have for example the trivial uniform solutions with zero energy, given by $\eta = \pm v$. Here though, we are interested in the non-trivial 'kink' solution,

$$\eta = v \tanh(my)$$

where $m = \sqrt{2\lambda}v$. The total energy is,

$$E = \int_{-\infty}^{\infty} dy \left\{ \frac{1}{2}\left(\frac{d\eta}{dy}\right)^2 + \lambda(\eta^2 - v^2)^2 \right\}$$

We can see from this that the energy is localised around $y = 0$. It is interesting to note that because the equation of motion is Lorentz-invariant, we can actually boost this solution to obtain a moving lump of energy; the kink is somewhat like a particle. For this reason it is sometimes referred to as a **soliton**[22].

This solution is called a 'topological defect', because even though it is not the lowest-energy solution, we can argue that it is completely stable; it cannot decay to one of the ground states $\eta = \pm v$. To see this, notice that for any finite amount of energy, the field must asymptotically assume one of its vaccum values. Decay to a ground state would require the asymptotic value in one direction to change from one vacuum value to the other, but there is no continuous way for this to happen. Therefore we say the kink is **topologically stable**.

B.2 The general case

The case of the kink is actually a somewhat trivial example, because the vacuum manifold is disconnected. More interesting cases occur in theories with more spatial dimensions and a more complicated vacuum manifold. Usually this involves having a multiplet of scalar fields in a non-trivial representation of some Lie group.

[22] Technically, the kink does not satisfy the criteria of a soliton, but of a slightly weaker object called a solitary wave.

Denote our scalar field multiplet by Φ, and the vacuum manifold by M. The ground states of the theory occur when Φ uniformly assumes some value $p \in M$.

If the theory lives in $n+1$ dimensions, the spatial part is topologically $\mathbb{R}^n$, so its boundary is S^{n-1}. Therefore boundary conditions for a finite-energy solution correspond to a map $f : S^{n-1} \to M$. If the $(n-1)^{th}$ homotopy group of M is non-trivial, there arises the possibility that f is homotopically non-trivial. In this case, the solution cannot be uniform, and thus will have non-zero energy. But similarly to the kink, it cannot decay to the ground state.

To have finite energy at all times, the asymptotic values of Φ must always lie in M. Continuously deforming these from the configuration given by f to a constant would amount to a homotopy between f and a constant map, contradicting the fact that f is homotopically non-trivial. Therefore any such solution will be topologically stable.

C Numerical techniques

The main results of the project, presented in section 7, were only able to be obtained by solving the differential equations numerically. This was achieved using C code, which was modified and expanded from code written by Damien George to solve a similar problem. A significant amount of the author's time was spent on the numerical aspects of the work, and as such it seems appropriate to give a very brief description of how it was done.

C.1 The relaxation method

The first thing to do was actually find solutions to the equations (5). Both of these equations are of the form $f'' = S(f)$ for some function S, and this makes them quite simple to deal with numerically.

When finding solutions numerically, functions are approximated by their value at a finite set of points $\{y_n\}$. So the function f is represented by the array $\{f(y_1), f(y_2), ..., f(y_N)\}$. This means we have to find discretised versions of derivative operators. For simplicity, we work with a fixed step size, so that $|y_n - y_{n-1}| = h$ for some fixed h and all values of n. There are two obvious choices for the first derivative operator,

$$f'(y_n) \simeq \frac{f(y_{n+1}) - f(y_n)}{h} \tag{11}$$

or

$$f'(y_n) \simeq \frac{f(y_n) - f(y_{n-1})}{h} \tag{12}$$

(11) is an approximation for the upper derivative, and (12) is an approximation for the lower derivative. For a differentiable function, these two agree in the limit $h \to 0$. We need a second derivative though, so we iterate this definition to obtain,

$$\begin{aligned} f''(y_n) &\simeq \frac{f'(y_{n+1}) - f'(y_n)}{h} \\ &\simeq \frac{f(y_{n+1}) - 2f(y_n) + f(y_{n-1})}{h^2} \end{aligned} \tag{13}$$

where in going from the first line to the second, we have substituted (12) for the first term, and (11) for the second. This is a well-known approximation for the second derivative. Where we go from here depends on the boundary or initial conditions of the problem. In our case, we have Dirichlet boundary conditions at $y = \pm\infty$, which means here that we will enter the values of $f(y_1)$ and $f(y_N)$ by hand. So we have the values of f at each end, and need to 'fill in' the middle. To do this, we re-arrange (13) to get $f(y_n)$ in terms of the neighbouring values,

$$\begin{aligned} f(y_n) &= \frac{1}{2}\Big(f(y_{n+1}) + f(y_{n-1}) - h^2 f''(y_n)\Big) \\ &= \frac{1}{2}\Big(f(y_{n+1}) + f(y_{n-1}) - h^2 S(f(y_n))\Big) \end{aligned} \tag{14}$$

where we have used the fact that f solves $f'' = S(f)$. This expression forms the basis of our method of solution. We put in our Dirichlet boundary conditions, and also a trial expression for f on the interior (most of the time, this was just taken to be a constant). We then go through and set each $f(y_n)$ equal to the above expression. Iterating this procedure, f converges to a solution. This is known as the **relaxation method**, because f 'relaxes' to a configuration that solves the equation. In reality, we set some small value for the acceptable 'error', and once the maximum value of $\Delta f(y_n)/f(y_n)$ drops below this number, terminate the program and read out what we hope is very close to a solution!

Notes
One thing to note is that with non-linear differential equations, such as those encountered in this project, solutions are not necessarily unique, even once boundary conditions are specified. Therefore different expressions for the trial function can conceivably lead to different solutions of the equation. For example, one solution for the adjoint scalar in section 7 is identically zero. Obviously this is not the solution we wanted, so had to use a non-zero trial function to try to find a less trivial possibility.

Once a solution is thought to have been found, we have to ensure it is a genuine solution by checking it is insensitive to the details of the numerics. As such, each time a solution was found, the code was run again multiple times with smaller error tolerance, or smaller step size, or larger maximum value of y (we had of course to find solutions on a finite domain!). An approximation to a genuine solution will not depend significantly on such numerical parameters. All the solutions presented in section 7 passed these tests.

C.2 Finding eigenvalues

We demonstrated perturbative stability of the solutions we found in section 7 by showing that all perturbations resulted in oscillatory behaviour. This involved finding the eigenvalues of a second order linear differential equation.

Differential operators are linear, so if f is being represented by a (large!) vector $(f(y_1), f(y_2), ..., f(y_N))$, the operator d^2/dy^2 will take the form of a matrix. We need to know what this matrix is. Returning to our approximation,

$$f''(y_n) \simeq \frac{f(y_{n+1}) - 2f(y_n) + f(y_{n-1})}{h^2}$$

we can easily see what this matrix needs to be:

$$d^2/dy^2 \longrightarrow \frac{1}{h^2}\begin{pmatrix} \ddots & & & & \\ 1 & -2 & 1 & 0 & \\ 0 & 1 & -2 & 1 & 0 \\ & 0 & 1 & -2 & 1 \\ & & & & \ddots \end{pmatrix}$$

The problem is thus reduced to finding the eigenvalues of a very large matrix. The algorithms for doing this were taken from 'Numerical Recipes in C'.

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